The Isomorphism Theorems

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Theorem (First Isomorphism Theorem)

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Corollary

Let $\phi : G \to H$ *be a homomorphism of groups. Then,*

1. ϕ *is injective if and only if* $\ker(\phi) = \{1_G\}$.
2. $[G : \ker(\phi)] = |\phi(G)|$.

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Theorem (Second or Diamond Isomorphism Theorem)

Suppose that $A, B \le G$ *with* $A \le N_G(B)$. *Then,* $AB \le G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ *and* $AB/B \cong A/(A \cap B)$.

Theorem (Third Isomorphism Theorems)

Let G be a group and let $H, K \trianglelefteq G$ with $H \leq K$. Then $K/H \trianglelefteq G/H$ and

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Theorem (Fourth Isomorphism Theorem)

Let G be a group and let $N \trianglelefteq G$. Then there is a bijection from the set of subgroups $A \leq G$ which contain N onto the set of subgroups $A/N \leq G/N$. In particular, every subgroup of G/N is of the form A/N for some $N \leq A \leq G$. This bijection has the following properties.

1. $A \leq B$ *if and only if* $A/N \leq B/N$,
2. *if* $A \leq B$ *then,* $[B : A] = [B/N : A/N]$,
3. $< A, B > /N =< A/N, B/N >$,
4. $(A \cap B)/N = A/N \cap B/N$, *and*
5. $A \trianglelefteq G$ *if and only if* $A/N \trianglelefteq G/N$.

Note

Suppose $N \trianglelefteq G$ and that $\Phi : G \to H$ is a homomorphism then the map $\phi : G/N \to H$ is well-defined if and only if $N \leq \ker(\Phi)$. In this case, we say that the map Φ factors through N and that ϕ is the induced homomorphism on G/N.