Composition Series and the Hölder Program

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Remark

The fourth isomorphism theorem illustrates how an understanding of the normal subgroups of a given group can help us to understand the group itself better. The proof of the proposition below which is a special case of Cauchy's theorem illustrates how such information can be used inductively to prove theorems.

Proposition

If G is a finite Abelian group and p is a prime dividing $|G|$, then G contains an element of order p.

Definition

A group G is called simple if $|G| > 1$ and the only normal subgroups of G are $\{1_G\}$ and G.

Note

1. If G is Abelian and simple then $G = Z_p$.
2. There are both infinite and finite non-abelian groups which are simple. The smallest non-Abelian simple group is A_5 which has order 60.

Definition

In a group G a sequence of subgroups

$$1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1} \leq N_k = G$$

is called a composition series for G if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is simple for $0 \leq i \leq k-1$.
If the above sequence is a composition series the quotient groups N_{i+1}/N_i are called composition factors of G.

Theorem (Jordan-Hölder)

Let $1 \neq G$ be a finite group. Then,

1. *G has a composition series*
2. *The (Jordan Hölder) composition factors in any composition series of G are unique up to isomorphism and rearrangement.*

We will proceed by induction on $|G|$.

Base Case: The cases $|G| = 2, 3$ are clear since in these cases G is simple and the only decomposition series is $1 \trianglelefteq G$.

Induction Hypothesis We assume that the theorem holds for all groups M with $2 \leq |M| < n$.

Induction Step Suppose that $|G| = n$.

There are 2 cases to consider: either G is simple or it is not.

Case 1: *G is simple.*

In this case, the only possible decomposition series is $1 \trianglelefteq G$ and so the statement of the theorem holds.

Case 2: *G is not simple.*

Existence of a decomposition series Since G is not simple, $\exists 1 \neq N \neq G$ with $N \trianglelefteq G$.
Thus, $|N|, |G/N| < |G|$. So, by our inductive hypothesis we see that both N and G/N have decomposition series.
Note that by the 4th isomorphism theorem the decomposition series for G/N can be lifted to a nested chain of normal subgroups of G which complete the decomposition series of N to one for G.
So, we see that G has a decomposition series.

Uniqueness of Decomposition Factors up to Isomorphism and rearrangement

Suppose that

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G \qquad \text{and}$$
$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_{\ell-1} \trianglelefteq K_\ell = G$$

are decomposition series for G.
Let $H = H_{k-1}$; $K = K_{\ell-1}$; $L = H \cap K$.
We have 2 subcases to consider: $H = K$ and $H \neq K$.

Case 2.1$H = K$
Since $|H|, |K| < |G|$ the uniqueness of the decomposition factors follows from our induction hypothesis and the fact that the first factors in both series are the same.
Case 2.2$H \neq K$.
In this case, we have that $H, K \trianglelefteq G$, and thus we also have $H, K \trianglelefteq HK \trianglelefteq G$.
Since the decomposition factors G/H and G/K are simple and since $H \neq K$, we conclude that $HK = G$.

Proof of Jordan-Hölder Theorem continued

Consider the following diagram.

$$
\begin{array}{ccccccc}
 & & & G & & & \\
 & & & \| & & & \\
 & & & HK & & & \\
 & & / & & \backslash & & \\
 & H & & & & K & \\
 / & & \backslash & & / & & \backslash \\
H_{k-2} & & & L & & & K_{\ell-2} \\
\vdots & & & & & & \vdots \\
H_0 & & & & & & K_0
\end{array}
$$

Note that by 2nd isomorphism theorem $G/H \cong K/L$ and $G/K \cong H/L$.
Also, since $|L| < |G|$, L has a decomposition series....

Proof of Jordan-Hölder Theorem continued

$$
\begin{tikzcd}[column sep=small]
 & & G \arrow[d, equal] & & \\
 & & HK \arrow[dl, dash] \arrow[dr, dash] & & \\
 & H \arrow[dl, dash] \arrow[dr, dash] & & K \arrow[dl, dash] \arrow[dr, dash] & \\
H_{k-2} \arrow[d, dash] & & L \arrow[d, dash] & & K_{\ell-2} \arrow[d, dash] \\
\vdots & & L_{t-1} & & \vdots \\
\vdots & & \vdots & & \vdots \\
H_0 & & L_0 & & K_0
\end{tikzcd}
$$

By our induction hypothesis and the fact that $|H|, |K| < |G|$, we can conclude that $k - 2 = t = \ell - 2 \Rightarrow k = \ell$.

Proof of Jordan-Hölder Theorem continued

$$
\begin{array}{ccccccccc}
 & & & & G & & & & \\
 & & & / & & \backslash & & & \\
 & & H & & & & K & & \\
 & / & & \backslash & & / & & \backslash & \\
H_{k-2} & & & & L & & & & K_{\ell-2} \\
| & & & & | & & & & | \\
\vdots & & & & L_{t-1} & & & & \vdots \\
\vdots & & & & \vdots & & & & \vdots \\
H_0 & & & & L_0 & & & & K_0
\end{array}
$$

By the 2nd isomorphism theorem and induction, we have

$$
\begin{aligned}
\{H_j/H_{j-1}\}_{j=1}^{k} &= \{G/H\} \cup \{H_j/H_{j-1}\}_{j=1}^{k-1} \\
&= \{G/H\} \cup \{H/L\} \cup \{L_j/L_{j-1}\}_{j=1}^{t} \\
&= \{K/L\} \cup \{G/K\} \cup \{L_j/L_{j-1}\}_{j=1}^{t} \\
&= \{G/K\} \cup \{K_j/K_{j-1}\}_{j=1}^{k-1} = \{K_j/K_{j-1}\}_{j=1}^{k}
\end{aligned}
$$

The Hölder Program

1. Classify all finite simple groups.
2. Find all ways of combining simple groups to form other groups.

Theorem

There is a list consisting of 18 (infinite) families of simple groups and 26 simple groups not belonging to these families (the sporadic simple groups) such that every finite simple group is isomorphic to one of the groups in the list.

Note

Some examples of families of simple groups are as follows.

1. $\{Z_p \quad | \quad p \text{ is prime.}\}$.
2. $$\{\mathbb{S}L_n(\mathbb{F}) \quad / \quad Z(\mathbb{S}L_n(\mathbb{F})) \quad | \quad 2 \leq n \in \mathbb{Z}; \mathbb{F} \text{ is a finite field}\} \setminus \{\mathbb{S}L_2(\mathbb{F}_2), \mathbb{S}L_2(\mathbb{F}_3)\}$$
3. $\{A_n \trianglelefteq S_n \quad | \quad n \geq 5\}$.
4. The largest of the sporadic simple groups is the "Monster" which has order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 =$ 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000.

Definition

A group G is solvable if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is Abelian for $0 \leq i \leq (s-1)$.

Note

If N and G/N are solvable then G is solvable.