Transpositions and the Alternating Group

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Definition

In S_n any 2-cycle is called a transposition.

Note

1. Recall that every permutation can be written as a product of disjoint cycles.
2. $(a_1, \ldots, a_m) =$
   $(a_1, a_m)(a_1, a_{m-1})(a_1, a_{m-2}) \ldots (a_1, a_3)(a_1, a_2)$.
3. Every permutation can be written as a product of transpositions.

Definition

1. $\Delta_n = \prod_{1 \leq i < j \leq n}(x_i - x_j)$.
2. For $\sigma \in S_n$, define $\sigma(\Delta_n) = \prod_{1 \leq i < j \leq n}(x_{\sigma(i)} - x_{\sigma(j)})$.

Note

For all $\sigma \in S_n$, $\sigma(\Delta_n) = \pm\Delta_n$.

Definition

Suppose $\sigma \in S_n$. We define the following.

1. $\epsilon(\sigma) = \sigma(\Delta_n)/\Delta_n \quad \in \{\pm 1\}$.
2. $\epsilon(\sigma)$ is called the <u>sign</u> of σ.
3. σ is called an <u>even</u> permutation if $\epsilon(\sigma) = 1$ and an <u>odd</u> permutation if $\epsilon(\sigma) = -1$.

Proposition

The map $\epsilon : S_n \to \{\pm 1\}$ is a homomorphism.

Proposition

Transpositions are all odd permutations and ϵ is a surjective homomorphism.

Definition

The Alternating Group is defined as $A_n = \ker(\epsilon)$. That is, the Alternating group is the set of even permutations.

Note

Note that $S_n/A_n \cong \{\pm 1\}$. Thus $|A_n| = |S_n|/2 = \frac{n!}{2}$.

Fact

A_n is a non-Abelian simple group for all $n \geq 5$.