Group Actions and Permutations Representations

Kevin James

Recall

1. If G is a group acting on a set A, then $\forall g \in G$ we have a map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$.
2. The map $\phi : G \to S_A$ defined by $\phi(g) = \sigma_g$ is a homomorphism. The map ϕ is called the permutation representation associated to the group action of G on A.

Definition

1. The kernel of the action is $\{g \in G \quad | \quad g \cdot a = a, \forall a \in A\}$.
2. For each $a \in A$, the stabilizer of a in G is $G_a = \{g \in G \quad | \quad g \cdot a = a\}$.
3. An action is said to be faithful if its kernel is the identity.

Note

1. If G acts on A and ϕ is the induced permutation representation the the kernel of the action is $\ker(\phi)$.
2. So, g_1 and g_2 induce the same permutation on A if and only if $g_1 \ker(\phi) = g_2 \ker(\phi)$.
3. Thus the group action of G on A induces a faithful action of $G/\ker(\phi)$ on A.
4. $\ker(\phi) = \cap_{a \in A} G_a$.

Proposition

For any group G and any nonempty set A, there is a bijection between the actions of G on A and Hom(G, S_A) the homomorphisms of G into S_A.

Definition

If G is a group, a permutation representation of G is any homomorphism of G into S_A for some non-empty set A. We shall say that a given action of G on A affords or induces the associated permutation representation of G.

Proposition

Let G be a group acting on the nonempty set A. The relation on A defined by

$$a \sim b \qquad \text{if and only if } a = g \cdot b \text{ for some } g \in G.$$

is an equivalence relation. For each $a \in A$, $\#[a] = [G : G_a]$.

Definition

Let G be a group acting on a nonempty set A.

1. The equivalence class $\{g \cdot a \quad | \quad g \in G\}$ is called the orbit of G containing a.
2. The action of G on A is called transitive if there is only one orbit.