Group Actions and Permutations Representations

Kevin James
Recall

1. If $G$ is a group acting on a set $A$, then $\forall g \in G$ we have a map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$.

2. The map $\phi : G \to S_A$ defined by $\phi(g) = \sigma_g$ is a homomorphism. The map $\phi$ is called the permutation representation associated to the group action of $G$ on $A$. 
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3. An action is said to be faithful if its kernel is the identity.
If $G$ acts on $A$ and $\phi$ is the induced permutation representation the the kernel of the action is $\ker(\phi)$. 

Thus the group action of $G$ on $A$ induces a faithful action of $G/\ker(\phi)$ on $A$. 

\[ \ker(\phi) = \bigcap_{a \in A} G^a. \]
1 If $G$ acts on $A$ and $\phi$ is the induced permutation representation the the kernel of the action is $\ker(\phi)$.

2 So, $g_1$ and $g_2$ induce the same permutation on $A$ if and only if $g_1 \ker(\phi) = g_2 \ker(\phi)$. 

Proposition

For any group $G$ and any nonempty set $A$, there is a bijection between the actions of $G$ on $A$ and $\text{Hom}(G, S^A)$ the homomorphisms of $G$ into $S^A$. 

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**Note**

1. If $G$ acts on $A$ and $\phi$ is the induced permutation representation the the kernel of the action is $\ker(\phi)$.
2. So, $g_1$ and $g_2$ induce the same permutation on $A$ if and only if $g_1 \ker(\phi) = g_2 \ker(\phi)$.
3. Thus the group action of $G$ on $A$ induces a faithful action of $G/\ker(\phi)$ on $A$.
4. $\ker(\phi) = \cap_{a \in A} G_a$.

**Proposition**

For any group $G$ and any nonempty set $A$, there is a bijection between the actions of $G$ on $A$ and $\text{Hom}(G, S_A)$ the homomorphisms of $G$ into $S_A$. 
If $G$ is a group, a permutation representation of $G$ is any homomorphism of $G$ into $S_A$ for some non-empty set $A$. We shall say that a given action of $G$ on $A$ affords or induces the associated permutation representation of $G$. Proposition: Let $G$ be a group acting on the nonempty set $A$. The relation on $A$ defined by $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$. is an equivalence relation. For each $a \in A$, $\# [a] = [G : G_a]$. 

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**Proposition**

Let $G$ be a group acting on the nonempty set $A$. The relation on $A$ defined by $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$ is an equivalence relation. For each $a \in A$, $\# [a] = |G : Ga|$. 

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**Definition**

If $G$ is a group, a **permutation representation of** $G$ is any homomorphism of $G$ into $S_A$ for some non-empty set $A$. We shall say that a given action of $G$ on $A$ affords or induces the associated permutation representation of $G$.

**Proposition**

Let $G$ be a group acting on the nonempty set $A$. The relation on $A$ defined by

$$a \sim b \quad \text{if and only if} \quad a = g \cdot b \quad \text{for some} \quad g \in G.$$  

is an equivalence relation. For each $a \in A$, $\#[a] = [G : G_a]$.  

**Definition**

Let $G$ be a group acting on a nonempty set $A$.

1. The equivalence class $\{g \cdot a \mid g \in G\}$ is called the orbit of $G$ containing $a$.
2. The action of $G$ on $A$ is called transitive if there is only one orbit.