Group Actions and Permutations Representations

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Recall

1. If G is a group acting on a set A, then $\forall g \in G$ we have a map $\sigma_g : A \to A$ defined by $\sigma_g(a) = g \cdot a$.
2. The map $\phi : G \to S_A$ defined by $\phi(g) = \sigma_g$ is a homomorphism. The map ϕ is called the permutation representation associated to the group action of G on A.

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3. An action is said to be faithful if its kernel is the identity.

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Proposition

For any group G and any nonempty set A, there is a bijection between the actions of G on A and $Hom(G, S_A)$ the homomorphisms of G into S_A.

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Proposition

Let G be a group acting on the nonempty set A. The relation on A defined by

$$a \sim b \qquad \text{if and only if } a = g \cdot b \text{ for some } g \in G.$$

is an equivalence relation. For each $a \in A$, $\#[a] = [G : G_a]$.

Definition

Let G be a group acting on a nonempty set A.

1. The equivalence class $\{g \cdot a \quad | \quad g \in G\}$ is called the <u>orbit</u> of G containing a.
2. The action of G on A is called <u>transitive</u> if there is only one orbit.