Groups acting on themselves by conjugation – The Class Equation

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Note

In this section we will consider the following action of G on itself which is called conjugation.

$$g \cdot a = gag^{-1}.$$

Note that this is a well-defined group action.

Definition

Two elements $a, b \in G$ are said to be conjugate if there is some $g \in G$ such that $a = gbg^{-1}$. That is, if they are in the same orbit of G acting on itself by conjugation.
The orbits of G acting on itself by conjugation are called conjugacy classes.

Note

1. The action of conjugation is NOT transitive. In fact, $[1_G] = \{1_G\}$.
2. For any $S \subseteq G$, we can define $gSg^{-1} = \{gsg^{-1} \mid g \in G\}$ and thus the action of conjugation can be extended to the power set 2^G of G.

Definition

Two subsets $T, S \subseteq G$ are said to be conjugate if $S = gTg^{-1}$ for some $g \in G$.

Proposition

The number of conjugates of a subset $S \subseteq G$ is given by $[G : N_G(S)]$ In particular, $\#\{gsg^{-1} \mid g \in G\} = [G : C_G(s)]$.

Theorem (Class Equation)

Let G be a finite group and let $g_1, \ldots, g_r$ be representatives of the distinct conjugacy classes of G not contained in $Z(G)$. Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

Note

All summands on the right-hand side are divisors of $|G|$.

Theorem

If p is prime and P is a group of prime power order p^α with $\alpha \geq 1$, then P has a nontrivial center.

Corollary

If $|P| = p^2$ for some prime p, then P is Abelian. More precisely P is isomorphic to Z_{p^2} or $Z_p \times Z_p$.

PROPOSITION

Let $\sigma, \tau \in S_n$ and suppose that
$\sigma = (a_{1,1}, a_{1,2}, \ldots, a_{1,k_1})(a_{2,1}, \ldots, a_{2,k_2}) \ldots (a_{m,q}, \ldots a_{m,k_m})$.
Then

$$\begin{aligned}\tau\sigma\tau^{-1} = (\tau(a_{1,1}), \tau(a_{1,2}), \ldots, \tau(a_{1,k_1})) \quad & (\tau(a_{2,1}), \ldots, \tau(a_{2,k_2})) \\ \ldots \quad & (\tau(a_{m,q}), \ldots \tau(a_{m,k_m})).\end{aligned}$$

DEFINITION

1. If $\sigma \in S_n$ is a product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including 1-cycles) then the above sequence of integers is called the <u>cycle type</u> of σ.
2. If $0 < n \in \mathbb{Z}$, a <u>partition</u> of n is any nondecreasing sequence of positive integers whose sum is n.

Proposition

Two elements of S_n are conjugate in S_n if and only if they have the same cycle type. The number of conjugacy classes of S_n is equal to the number of partitions of n.

Note

If $H \trianglelefteq G$ then H is a union of conjugacy classes of G.

Theorem

A_5 is a simple group.