Groups acting on themselves by conjugation – The Class Equation

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The orbits of G acting on itself by conjugation are called conjugacy classes.

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Definition

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Proposition

The number of conjugates of a subset $S \subseteq G$ is given by $[G : N_G(S)]$ In particular, $\#\{gsg^{-1} \mid g \in G\} = [G : C_G(s)]$.

Theorem (Class Equation)

Let G be a finite group and let $g_1, \ldots, g_r$ be representatives of the distinct conjugacy classes of G not contained in $Z(G)$. Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

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Note

All summands on the right-hand side are divisors of $|G|$.

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Corollary

If $|P| = p^2$ for some prime p, then P is Abelian. More precisely P is isomorphic to Z_{p^2} or $Z_p \times Z_p$.

Proposition

Let $\sigma, \tau \in S_n$ *and suppose that*
$\sigma = (a_{1,1}, a_{1,2}, \ldots, a_{1,k_1})(a_{2,1}, \ldots, a_{2,k_2}) \ldots (a_{m,q}, \ldots a_{m,k_m})$.
Then

$$\begin{aligned}\tau\sigma\tau^{-1} = (\tau(a_{1,1}), \tau(a_{1,2}), \ldots, \tau(a_{1,k_1})) &\quad (\tau(a_{2,1}), \ldots, \tau(a_{2,k_2})) \\ \ldots &\quad (\tau(a_{m,q}), \ldots \tau(a_{m,k_m})).\end{aligned}$$

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Definition

1. If $\sigma \in S_n$ is a product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including 1-cycles) then the above sequence of integers is called the cycle type of σ.
2. If $0 < n \in \mathbb{Z}$, a partition of n is any nondecreasing sequence of positive integers whose sum is n.

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Theorem

A_5 is a simple group.