Automorphisms

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Definition

For G a group, an isomorphism from G to itself is called an automorphsim.

Fact

For a group G, the set $\mathrm{Aut}(G)$ of automorphisms of G is a group under composition of functions. In fact, $\mathrm{Aut}(G) \leq S_G$.

Proposition

Let $H \trianglelefteq G$. Then G acts by conjugation on H as automorphisms of H.
More specifically, the action of G on H by conjugation is defined for each $g \in G$ by $h \mapsto ghg^{-1}$ for all $h \in H$. For each $g \in G$, conjugation by g is an automorphism of H. The permutation representation afforded by this action is a homomorphism of G into $\mathrm{Aut}(H)$ with kernel $C_G(H)$.
In particular, $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

Corollary

If $K \leq G$ and $g \in G$, then $K \cong gKg^{-1}$. Conjugate elements and conjugate subgroups have the same order.

Corollary

If $H \leq G$ then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.
In particular, $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

Definition

Suppose that G is a group and that $g \in G$. Conjugation by g is called an inner automorphism of G and the subgroup of $\mathrm{Aut}(G)$ consisting of all inner automorphisms is denoted $\mathrm{Inn}(G)$.

Note

1. $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.
2. By the previous corollary, $\mathrm{Inn}(G) \cong G/Z(G)$.
3. If $H \trianglelefteq G$, then conjugation by $g \in G \setminus H$ yields an automorphism of H which may not be in $\mathrm{Inn}(H)$. (-e.g. consider $(1, 2, 3)$ in S_4 acting by conjugation on A_4.)

Definition

Suppose that G is a group. A subgroup $H \leq G$ is called <u>characteristic</u> in G denoted H char G, if $\forall \sigma \in \mathrm{Aut}(G)$, $\sigma(H) = H$.

Fact

1. *Characteristic subgroups are normal.*
2. *If H is the unique subgroup of G of a given order, then H is characteristic in G.*
3. *If K char H and $H \trianglelefteq G$, then $K \trianglelefteq G$.*