Sylow's Theorem

Kevin James

Definition

Let G be a group and let p be a prime.

1. A group of order p^α for some $\alpha \geq 1$ is called a <u>p-group</u>. Subgroups of G which are p-groups re called <u>p-subgroups</u>.
2. If $|G| = p^\alpha m$ where p is prime and $p \nmid m$, then a subgroup of order p^α is called a <u>Sylow p-subgroup</u> of G.
3. The set of Sylow p-subgroups of G will be denoted $\underline{Syl_p(G)}$ and the number of Sylow p-subgroups of G will be denoted $\underline{n_p(G)}$.

Theorem (Sylow's Theorem)

Let G be a group of order $p^\alpha m$ where p is prime and $p \nmid m$.

1. $Syl_p(G) \neq \emptyset$.
2. *If P is a Sylow p-subgroup of G and Q is any p-subgroup of G, then $\exists g \in G$ such that $Q \leq gPg^{-1}$. In particular, any two Sylow p-subgroups are conjugate in G.*
3. *We have that $n_p \equiv 1 \pmod{p}$. Also, $n_p = [G : N_G(P)]$ for any Sylow p-subgroup P. Thus $n_p | m$.*

Lemma

Let $P \in Syl_p(G)$. If Q is any p-subgroup of G, then $Q \cap N_G(P) = Q \cap P$.

Corollary

Let $P \in Syl_p(G)$. Then the following are equivalent.

1. *P is the unique Sylow p-subgroup of G (-i.e. $n_p = 1$)*
2. *$P \trianglelefteq G$.*
3. *$P char G$.*
4. *If $X \subseteq G$ has the property that $x \in X \Rightarrow |x| = p^m$ for some m, then $< X >$ is a p-group.*

Proposition

If $|G| = 60$ and if $n_5(G) > 1$ then G is simple.

Corollary

A_5 is simple.

Proposition

If G is a simple group of order 60, then $G \cong A_5$.