Direct Products

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Definition

1. Suppose that $G_1, \dots, G_n$ are groups with operations $*_1, *_2, \dots, *_n$. Then their direct product $G_1 \times \cdots \times G_n$ has a binary operation defined componentwise. That is,

$$(g_1, \dots, g_n)(h_1, \dots, h_n) = (g_1 *_1 h_1, \dots, g_n *_n h_n).$$

2. Suppose that $G_1, G_2, \dots$ are groups with operations $*_1, *_2, \dots$. Then the direct product of these groups has an operation defined componentwise as above.

Proposition

If $G_1, \dots, G_n$ are groups, their direct product is a group of order $|G_1| \cdot |G_2| \cdot \dots \cdot |G_n|$.

Proposition

Let $G_1, G_2, \ldots, G_n$ be groups and let $G = G_1 \times \cdots \times G_n$ be their direct product.

1. *For each $1 \leq i \leq n$ we define*

$$\mathfrak{G}_i = \{(1_{G_1}, \ldots, 1_{G_{i-1}}, g, 1_{G_{i+1}}, \ldots, 1_{G_n}) \quad | \quad g \in G_i\}.$$

   We have that $G_i \cong \mathfrak{G}_i$ is a subgroup of G and identifying G_i with $\mathfrak{G}_i$, we have that $G_i \trianglelefteq G$ and that $G/G_i \cong G_1/G_i \times \cdots \times G_{i-1}/G_i \times G_{i+1}/G_i, \times \cdots \times G_n/G_i$.

2. *For each $1 \leq i \leq n$, define $\pi_i : G \to G_i$ by $\pi_i((g_1, \ldots, g_n)) = g_i$. Then π is a surjective homomorphism with $\ker = \{(g_1, \ldots, g_{i-1}, 1_{G_i}, g_{i+1}, \ldots, g_n) \quad | \quad g_j \in G_j\} \cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$.*

3. *Under these identifications if $x \in G_i$ and $y \in G_j$ with $i \neq j$, then $xy = yx$.*