Direct Products

Kevin James

Definition

1. Suppose that $G_1, \ldots, G_n$ are groups with operations $*_1, *_2, \ldots, *_n$. Then their direct product $G_1 \times \cdots \times G_n$ has a binary operation defined componentwise. That is,

$$(g_1, \ldots, g_n)(h_1, \ldots, h_n) = (g_1 *_1 h_1, \ldots, g_n *_n h_n).$$

2. Suppose that $G_1, G_2, \ldots$ are groups with operations $*_1, *_2, \ldots$ Then the direct product of these groups has an operation defined componentwise as above.

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Proposition

If $G_1, \ldots, G)n$ are groups, their direct product is a group of order $|G_1| \cdot |G_2| \ldots |G_n|$. Likewise if $G_1, G_2, \ldots$ are groups then their direct product is a group.

Proposition

Let $G_1, \ldots, G_n$ be groups and let $G = G_1 \times \cdots \times G_n$ be their direct product.

1. *For each $1 \leq i \leq n$, we define*

$$\mathcal{G}_i = \{(1_{G_1}, \ldots, 1_{G_{i-1}}, g, 1_{G_{i+1}}, \ldots 1_{G_n}) \quad | \quad g \in G_i\}.$$

   We have that $G_i \cong \mathcal{G}_i$ which is a subgroup of G and identifying G_i with $\mathcal{G}_i$, we have that $G_i \trianglelefteq G$ and that $G/G_i \cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$.

2. *For each $1 \leq i \leq n$, define $\pi_i : G \to G_i$ by $\pi((g_1, \ldots, g_n)) = g_i$. Then, π_i is a surjective homomorphism with*

$$\begin{aligned} \ker(\pi_i) &= \{(g_1, \ldots, g_{i-1}, 1_{G_i}, g_{i+1}, \ldots, g_n) \quad | \quad g_j \in G_j\} \\ &\cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n. \end{aligned}$$

3. *Under these identifications if $x \in G_i$ and $y \in G_j$ and $i \neq j$, then $xy = yx$.*