Recognizing Direct Products

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Definition

Let G be a group and $x, y \in G$ and $\emptyset \neq A, B \subseteq G$.

1. Define the commutator of x and y to be $[x, y] = x^{-1}y^{-1}xy$.
2. Define $[A, B] =< \{[a, b] \quad | \quad a \in A; b \in B\} >$.
3. Define the commutator subgroup of G to be $G' = [G, G]$

Proposition

Let G be a group and let $x, y \in G$ and $G \leq G$. Then,

1. $xy = yx[x, y]$. *Thus* $xy = yx \Leftrightarrow [x, y] = 1$.
2. $H \trianglelefteq G$ *if and only if* $[H, G] \leq H$.
3. $\sigma[x, y] = [\sigma(x), \sigma(y)]$ *fo any* $\sigma \in \mathrm{Aut}(G)$, G' *char* G *and* G/G' *is Abelian.*
4. G/G' *is the largest Abelian quotient of* G, *that is if* $H \trianglelefteq G$ *and* G/H *is Abelian then* $G' \leq H$. *Conversely if* $G' \leq H$, *then* $H \trianglelefteq G$ *and* G/H *is Abelian.*
5. *If* $\phi : G \to A$ *is any homomorphism of* G *into an Abelian group* A, *then* ϕ *factors through* G' *(-i.e.* $G' \leq \ker(\phi)$ *and we have a commutative diagram....).*

Proposition

Let G be a group and $H, K \leq G$. The number of distinct ways of expressing a given element of HK as hk where $h \in H$ and $k \in K$ is $|H \cap K|$. In particular, if $H \cap K = 1$, then each element of HK can be written uniquely as hk for some $h \in H$ and $k \in K$.

Theorem

Suppose that G is a group and $H, K \leq G$ such that

1. $H, K \trianglelefteq G$ *and,*
2. $H \cap K = 1$.

Then, $HK \cong H \times K$.

Definition

Suppose that G is a group and $H, K \trianglelefteq G$ with $H \cap K = 1$, we call HK the internal direct product of H and K.

Example

Let $0 < n \in \mathbb{Z}$ be odd. Then $D_{4n} \cong D_{2n} \times Z_2$. More precisely, if we take $H =< s, r^2 >$ and $K =< r^n >$. Then $H \cong D_{2n}$, $K \cong Z_2$ and $D_{4n} = HK \cong H \times K$.