Semidirect Products

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Definition

Suppose that H and K are groups and let $\phi : K \to \mathrm{Aut}(H)$ be a homomorphism. We may define an action of K on H as
$k \cdot h = \phi(k)(h)$.
Then we define the semi-direct product $H \rtimes_\phi K$ of H and K with respect to ϕ as follows. As a set $H \rtimes_\phi K = H \times K$. The group operation of $H \rtimes_\phi K$ is defined by

$$\begin{aligned}(h_1, k_1)(h_2, k_2) &= (h_1(k_1 \cdot h_1), k_1 k_2) \\ &= (h_1 \phi(k_1)(h_1), k_1 k_2).\end{aligned}$$

Theorem

Let H and K be groups and let $\phi : K \to \mathrm{Aut}(H)$ be a homomorphism. Then

1. $H \rtimes_\phi K$ *is a group of order* $|H||K|$.
2. *Let* $\tilde{H} = \{(h, 1_K) \mid h \in H\}$ *and* $\tilde{K} = \{(1_H, k) \mid k \in K\}$. *Then* $\tilde{H}, \tilde{K} \leq H \rtimes_\phi K$ *with* $\tilde{H} \cong H$ *and* $\tilde{K} \cong K$.
3. $H \trianglelefteq H \rtimes_\phi K$.
4. $H \cap K = 1$.
5. $HK = H \rtimes_\phi K$.
6. $\forall h \in H$ *and* $k \in K$, $khk^{-1} = k \cdot h = \phi(k)(h)$.

Proposition

Let H and K be groups and let $\phi : K \to \mathrm{Aut} H$ be a homomorphism. The following are equivalent.

1. *The identity map between $H \rtimes K$ and $H \times K$ is a group homomorphism (and hence isomorphism).*
2. *ϕ is the trivial homomorphism from K into $\mathrm{Aut}(H)$.*
3. *$K \trianglelefteq H \rtimes K$.*

Theorem

Suppose that G is a group and $H, K \leq G$ such that

1. *$H \trianglelefteq G$, and*
2. *$H \cap K = 1$.*

Let $\phi : K \to \mathrm{Aut}(H)$ be the homomorphism defined by $\phi(k)(h) = khk^{-1}$. Then, $HK \cong H \rtimes K$. In particular, if $G = HK$ with H and K satisfying (1) and (2) above then G is the semidirect product of H and K.

Definition

Let H be a subgroup of G. A subgroup K is called a complement of H in G if $G = HK$ and $H \cap K = 1$.