p-groups, Nilpotent groups and Solvable groups

Kevin James

Definition

A maximal subgroup of a group G is a proper subgroup $M \leq G$ such that there are no subgroups H with $M < H < G$.

Theorem (p-groups)

Let p be a prime and let P be agroup of order p^a with $a \geq 1$. Then,

1. $Z(P) \neq 1$.
2. *If $1 \neq H \trianglelefteq P$ then $H \cap Z(P) \neq 1$. In particular, every normal subgroup of order p is contained in the center.*
3. *If $H \trianglelefteq P$, then for each $p^b \| H|$, $\exists K \leq H$ such that $K \trianglelefteq P$. In particular, P has a normal subgroup of order p^b for $1 \leq b \leq a$.*
4. *If $H < P$ then $H < N_P(H)$.*
5. *Every maximal subgroup of P is of index p and is normal in P.*

Definition

1. For G a group we inductively define:

$$Z_0(G) = 1, \qquad Z_1(G) = Z(G)$$

   and $Z_{i+1}(G)$ is the subgroup of G containing $Z_i(G)$ such that

$$Z_{i+1}(G)/Z_i(G) = Z\left(G/Z_i(G)\right)$$

   The chain $Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$ is called the upper central series of G.

2. A group G is called nilpotent if $Z_c(G) = G$ for some $c \in \mathbb{Z}$. The smallest such c is called the nilpotence class of G.

Note

1. One can show (for homework) that $Z_i(G)$ char G, $\forall i$.
2. If $1 \neq G$ is Abelian then G is nilpotent of class 1.
3. For finite groups, the upper central series must stabilize.
4. If two terms of the upper central series are the same then the series stabilize form that point onward.

Proposition

Let p be aprime and let P be a group of order p^a. Then P is nilpotent of nilpotence class at most $(a-1)$.

Theorem

Let G be a finite group and let $p_1, p_2, \dots, p_s$ be the distince primes dividing $|G|$ and let $P_i \in Syl_{p_i}(G)$, $1 \leq i \leq s$. Then the following are equivalent

1. *G is nilpotent.*
2. *If $H < G$ then $H < N_G(H)$.*
3. *$P_i \trianglelefteq G$ for $1 \leq i \leq s$.*
4. *$G \cong P_1 \times P_2 \times \cdots \times P_s$.*

Corollary

A finite Abelian group is the direct product of its Sylow subgroups.

Proposition

If G is a finite group such that for all positive integers n dividing its order, G contains at most n elements x satisfying $x^n = 1$, then G is cyclic.

Proposition (Frattini's Argument)

Let G be a finite group, let H be a normal subgroup of G and let P be a Sylow p-subgroup of H. Then $G = HN_G(P)$ and $[G : H]$ divides $|N_G(P)|$.

Proposition

A finite group is nilpotent if and only if every maximal subgroup is normal.

Definition

For any group G we inductively define:

$$G^0 = G, \qquad G^1 = [G, G], \quad \text{and} \quad G^{i+1} = [G, G^i].$$

Then chain of groups $G^0 \geq G^1 \geq \dots$ is called the lower central series of G.

Theorem

A group G is nilpotent if and only if $G^n = 1$ for some $n \geq 0$. More precisely, G is nilpotent of class c if and only if c is the smallest nonnegative integer such that $G^c = 1$. If G is nilpotent of class c then

$$Z_i(G) \leq G^{c-i-1} \leq Z_{i+1}(G) \qquad 0 \leq i \leq c-1.$$

Recall

A group G is said to be solvable if there is a series $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is Abelian.

Definition

For any group G we inductively define:

$$G^{(0)} = G, G^{(1)} = [G, G], \text{ and } G^{(i+1)} = [G^{(i)}, G^{(i)}] \text{ for all } i \geq 1.$$

This series of groups is called the derived or commutator series of G.

Theorem

A group G is solvable if and only if $G^{(n)} = 1$ for some $n \geq 0$.

Definition

It G is solvable, the smallest nonnetative n for which $G^{(n)} = 1$ is called the solvable length of G.

Proposition

Let G and K be groups, let $H \leq G$ and let $\phi : G \to K$ be a surjective homomorphism.

1. $H^{(i)} \leq G^{(i)}, \forall i \geq 0$. *In particular, if G is solvable then so is H and the solvable length of H is less than or equal to the solvable length of G.*
2. $\phi(G^{(i)}) = K^{(i)}$. *In particular, homomorphic images of quotient groups of solvable groups are solvable of solvable length less than or equal to that of the domain group.*
3. *If $N \trianglelefteq G$ and both N and G/N are solvable then so is G.*

Theorem

Burnside *If $|G| = p^a q^b$ for some primes p and q, then G is solvable.*

Philip Hall *If for every prime p dividing $|G|$, we write $|G| = p^a m$ with $(p, m) = 1$, and G has a subgroup of order m, then G is solvable.*

Feit-Thompson *If $|G|$ is odd then G is solvable.*

Thompson *If for every pair of elements $x, y \in G$, $< x, y >$ is solvable then so is G.*