Polynomial Rings, Matrix Rings and Group Rings

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Definition

Suppose that R is a ring. We define the ring of polynomials in the variable x as

$$R[x] = \{\sum_{n=0}^{d} a_n x^n \quad | \quad a_n \in R\}.$$

Addition and multiplication are defined as follows

- $\sum_{n=0}^{d_1} a_n x^n + \sum_{n=0}^{d_2} b_n x^n = \sum_{n=0}^{\max(d_1,d_2)} (a_n + b_n) x^n$.
- $$\begin{aligned} \left(\sum_{n=0}^{d_1} a_n x^n\right) \times \left(\sum_{n=0}^{d_2} b_n x^n\right) &= \sum_{n=0}^{d_1} \sum_{m=0}^{d_2} a_n b_m x^{n+m} \\ &= \sum_{n=0}^{d_1+d_2} \left(\sum_{j=0}^{n} a_i b_{n-i}\right) x^n \end{aligned}$$

Note

1. $R \hookrightarrow R[x]$. This copy of R in $R[x]$ is called the constant polynomials.
2. $R[x]$ is a ring with $0_{R[x]} = 0_R$.
3. If R is commutative then so is $R[x]$.
4. if R has an identity 1_R then $R[x]$ has an identity $1_{R[x]} = 1_R$.
5. If $f(x) = \sum_{n=0}^{d}$ and $a_d \neq 0_R$, then d is said to be the <u>degree</u> of $f(x)$ and a_d is said to be the leading coefficient. We will leave the degree of the 0 polynomial undefined but take the leading coefficient to be 0_R.

Proposition

Let R be an integral domain and let $p(x), q(x) \in R[x]$ be nonzero. Then,

1. $\deg(pq) = \deg(p) + \deg(q)$.
2. *the units of $R[x]$ are jus the units of R.*
3. *$R[x]$ is an integral domain.*

Definition

Suppose that R is a ring and that $0 \leq n \in \mathbb{Z}$. Then we define $M_n(R)$ to be the set of $n \times n$ matrices with coefficients in R. We define addition component wise as usual and define multiplication as for $M_n(\mathbb{R})$.

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NOTE

1. If R is nontrivial, then $M_n(R)$ is non-commutative.
2. If R is nontrivial, then $M_n(R)$ has zero divisors.
3. Note that $R \hookrightarrow M_n(R)$ as scalar matrices.
4. The scalar matrices commute if and only if R is commutative.
5. If R has an identity, then the matrix with 1_R in each diagonal entry and 0_R elsewhere is an identity for $M_n(R)$.
6. In the case that R has an identity, we define $\mathbb{G}L_n(R) = M_n(R)^*$

Definition

Suppose that R is a commutative ring with identity and that $G = \{g_1, \ldots, g_n\}$ is any finite group with group operation written multiplicatively. Define the group ring RG of G with coefficients in R as follows

$$RG = \left\{ \sum_{j=1}^{n} a_j g_j \quad | \quad a_j \in R; g_j \in G \right\},$$

where the sums are formal sums.

Addition is defined componentwise:

$\sum_{j=1}^{n} a_j g_j + \sum_{j=1}^{n} b_j g_j = \sum_{j=1}^{n} (a_j + b_j) g_j.$

Multiplication: We define $(ag_i)(bg_j) = (ab)(g_i g_j)$ where the first product is in R and the second is in G.

We extend the multiplication to RG as

$$\left(\sum_{j=1}^{n} a_j g_j \right) * \left(\sum_{j=1}^{n} b_j g_j \right) = \sum_{j=1}^{n} \left(\sum_{g_m g_n = g_j} a_m b_n \right) g_j$$

Note

1. If R and G are as above then RG is a ring.
2. It is not necessary for R to be commutative.
3. RG is commutative if and only if R and G are.
4. $R \hookrightarrow RG$.
5. The elements of R commute with the elements of RG assuming commutativity of R.
6. $1_{RG} = 1_R 1_G$.