Ring Homomorphisms and Quotient Rings

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Definition

Let R and S be rings.

1. A ring homomorphism is a map $\phi : R \to S$ satisfying
    1. $\phi(a+b) = \phi(a) + \phi(b)$ for all $a, b \in R$.
    2. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
2. The kernel $\ker(\phi)$ of ϕ is defined as $\ker(\phi) = \{r \in R \mid \phi(r) = 0_S\}$.
3. A bijective ring homomorphism is called a ring isomorphism.

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Proposition

Let R and S be rings and let $\phi : R \to S$ be a homomorphsim.

1. *The imabe of ϕ is a subring of S.*
2. $\ker(\phi)$ *is a subring of R with the additional property that for all $r \in \ker(\phi)$ and $a \in R$, $ra, ar \in \ker(\phi)$.*

Definition

Let R be a ring, let $I \subseteq R$ and let $r \in R$.

1. $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$.
2. A subset I of R is a left ideal or R if
    1. I is a subring of R, and
    2. $rI \subseteq I$ for all $r \in R$.
3. A subset I of R is a right ideal or R if
    1. I is a subring of R, and
    2. $Ir \subseteq I$ for all $r \in R$.
4. A subset I that is both a left ideal and right ideal is called an ideal. In this case, we write $I \leq R$.

Proposition

Let R be a ring and let $I \trianglelefteq R$. Then the quotient group R/I is a ring under the addition and multiplication operations

$$(r+I)+(s+I)=(r+s)+I, \quad (r+I)(s+I)=(rs)+I,$$

for all $r, s \in R$. Conversely if I is any subgroup sh that the above operation are well-defined, then $I \trianglelefteq R$.

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Definition

If R is a ring and $I \trianglelefteq R$, then R/I with operations as in the above proposition is called the quotient ring of R by I.

Theorem

1. *(The First Isomorphism Theorem) If $\phi : R \to S$ is a homomorphism of rings, then $\ker(\phi) \trianglelefteq R$ and $\phi(R)$ is a subring of S and $R/\ker(\phi) \cong \phi(R)$.*
2. *If I is any ideal of R, then the map $\pi : R \to R/I$ defined by $\pi(r) = r + I$ is a surjective ring homomorphism with $\ker(\pi) = I$. (π is called the natural projection of R onto R/I.)*