Rings of Fractions

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Theorem

Let R be a commutative ring. Let $\emptyset \neq D \subseteq R$ such that $0_R \notin D$ and such that D contains no zero divisors and is closed under multiplication. Then there is a commutative ring Q with identity such that Q contains R as a subring and every element of D is a unit in Q. The ring Q has the following additional properties.

1. *Every element of Q is of the form rd^{-1} for some $r \in R$ and $d \in D$. In particular, if $D = R \setminus \{0_R\}$ then Q is a field.*
2. *The ring Q is the smallest ring containing R in which all elements of D become units in the following sense. Let S be any commutative ring with identity and let $\phi : R \to S$ be any injective ring homomorphism such that $\phi(d)$ is a unit in S, $\forall d \in D$. Then there is an injective homomorphism $\Phi : Q \to S$ such that $\Phi|_R = \phi$. In other words, any ring containing an isomorphic copy of R in which all elements of D are units must also contain an isomorphic copy of Q.*

Definition

Let R, D and Q be as in the theorem.

1. The ring Q is called the ring of fractions of D with respect to R and is denoted $D^{-1}R$.
2. If R is an integral domain and $D = R \setminus \{0_R\}$, Q is called the field of fractions or quotient field of R.

Definition

If A is a subset of a field F, then the intersection of all subfields of F containing A is a subfield of F and is called the subfield generated by A.

Corollary

Let R be an integral domain and let Q be the quotient field of R. If a field F contains a subring R' isomorphic to R, then the subfield of F generated by R' is isomorphic to Q.