# Principal Ideal Domains

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### Definition

Suppose that $R$ is an integral domain. If all ideals in $R$ are principal then $R$ is called a Principal Ideal Domain.

### Note

In the last section we proved that every Euclidean Domain is a Principal Ideal Domain.

### Proposition

*Suppose that $R$ is a PID and that $a, b \in R$ are non-zero elements. Suppose that $(d) = (a, b)$. Then,*

1. *$d$ is a gcd of $a$ and $b$.*
2. *$d$ is a $R$-linear combination of $a$ and $b$.*
3. *$d$ is unique up to multiplication by a unit of $R$.*

## Proposition

*Every nonzero prime ideal in a PID is maximal.*

## Corollary

*If $R$ is any commutative ring such that $R[x]$ is a PID (or Euclidean Domain) then, $R$ is a field.*

## Definition

Define $N$ to be a Dedekind-Hasse norm if $N$ is a positive norm and if for all nonzero $a, b \in R$, either $a \in (b)$ or there is a nonzero element $x \in (a, b)$ with $N(x) < N(b)$. That is, either $b|a$ or there is $s, t \in R$ such that $0 < N(sa - tb) < N(b)$.

## Note

$R$ is Euclidean with respect to the positive norm $N$ if it satisfies the Dedekind-Hasse property with $s = 1_R$.

## Proposition

*The integral domain $R$ is a PID if and only if it has a Dedekind-Hasse norm.*

## Example

$R = \mathbb{Z}\left[\frac{(1+\sqrt{-19})}{2}\right]$ has a Dedekind-Hasse norm and is thus a PID although we saw at the end of the last section that it is not a Euclidean Domain.