Principal Ideal Domains

Kevin James

Definition

Suppose that R is an integral domain. If all ideals in R are principal then R is called a Principal Ideal Domain.

Definition

Suppose that R is an integral domain. If all ideals in R are principal then R is called a Principal Ideal Domain.

Note

In the last section we proved that every Euclidean Domain is a Principal Ideal Domain.

Definition

Suppose that R is an integral domain. If all ideals in R are principal then R is called a Principal Ideal Domain.

Note

In the last section we proved that every Euclidean Domain is a Principal Ideal Domain.

Proposition

Suppose that R is a PID and that $a, b \in R$ are non-zero elements. Suppose that $(d) = (a, b)$. Then,

1. *d is a gcd of a and b.*
2. *d is a R-linear combination of a and b.*
3. *d is unique up to multiplication by a unit of R.*

Proposition

Every nonzero prime ideal in a PID is maximal.

Proposition

Every nonzero prime ideal in a PID is maximal.

Corollary

If R is any commutative ring such that $R[x]$ is a PID (or Euclidean Domain) then, R is a field.

Definition

Define N to be a Dedekind-Hasse norm if N is a positive norm and if for all nonzero $a, b \in R$, either $a \in (b)$ or there is a nonzero element $x \in (a, b)$ with $N(x) < N(b)$. That is, either $b|a$ or there is $s, t \in R$ such that $0 < N(sa - tb) < N(b)$.

Definition

Define N to be a Dedekind-Hasse norm if N is a positive norm and if for all nonzero $a, b \in R$, either $a \in (b)$ or there is a nonzero element $x \in (a, b)$ with $N(x) < N(b)$. That is, either $b|a$ or there is $s, t \in R$ such that $0 < N(sa - tb) < N(b)$.

Note

R is Euclidean with respect to the positive norm N if it satisfies the Dedekind-Hasse property with $s = 1_R$.

Definition

Define N to be a Dedekind-Hasse norm if N is a positive norm and if for all nonzero $a, b \in R$, either $a \in (b)$ or there is a nonzero element $x \in (a, b)$ with $N(x) < N(b)$. That is, either $b | a$ or there is $s, t \in R$ such that $0 < N(sa - tb) < N(b)$.

Note

R is Euclidean with respect to the positive norm N if it satisfies the Dedekind-Hasse property with $s = 1_R$.

Proposition

The integral domain R is a PID if and only if it has a Dedekind-Hasse norm.

Definition

Define N to be a Dedekind-Hasse norm if N is a positive norm and if for all nonzero $a, b \in R$, either $a \in (b)$ or there is a nonzero element $x \in (a, b)$ with $N(x) < N(b)$. That is, either $b|a$ or there is $s, t \in R$ such that $0 < N(sa - tb) < N(b)$.

Note

R is Euclidean with respect to the positive norm N if it satisfies the Dedekind-Hasse property with $s = 1_R$.

Proposition

The integral domain R is a PID if and only if it has a Dedekind-Hasse norm.

Example

$R = \mathbb{Z}\left[\frac{(1+\sqrt{-19})}{2}\right]$ has a Dedekind-Hasse norm and is thus a PID although we saw at the end of the last section that it is not a Euclidean Domain.