Polynomial Rings especially over Fields

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Proposition

Let R be an integral domain. Then,

1. *If $p(x), q(x) \in R[x]$ are nonzero, then* $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$.
2. $(R[x])^\times = R^\times$.
3. *$R[x]$ is an integral domain.*

Proposition

Suppose that $I \trianglelefteq R$ and let $(I) = I[x]$ denote the ideal of $R[x]$ generated by I. Then,

$$R[x]/(I) \cong (R/I)[x].$$

In particular, if I is a prime ideal of R, then (I) is a prime ideal of $R[x]$.

Definition

The polynomial ring in n variables with coefficients in R is defined inductively for $n \geq 2$ as

$$R[x_1, x_2, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x_n].$$

Note

1. For $f \in R[x_1, x_2, \ldots, x_n]$, f can be expressed as a finite sum of the form $f(x_1, x_2, \ldots, x_n) = \sum_{0 \leq m_1, m_2, \ldots, m_n} a_{m_1, m_2, \ldots, m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where the $a_{\vec{m}} \in R$.
2. We define degree as follows:
   1. $\deg_i(a_{m_1, m_2, \ldots, m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) = m_i$.
   2. $\deg(a_{m_1, m_2, \ldots, m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) = max_{1 \leq i \leq n} (\deg_i(a_{m_1, m_2, \ldots, m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}))$.
   3. $\deg(f) = max_{\vec{m}}(\deg(a_{\vec{m}} x^{\vec{m}}))$.

Example

$\deg(x^2y + xy^2) = 3.$

Theorem

Let F be a field. The polynomail ring $F[x]$ is a Euclidean Domain. Specifically if $a(x), b(x) \in F[x]$ with $0 \neq b(x)$, then there is a unique $q(x), r(x) \in F[x]$ such that

$$a(x) = b(x)q(x) + r(x) \text{ with } r(x) = 0 \text{ or } \deg(r(x)) < \deg(b(x)).$$

Corollary

If F is a field, then $F[x]$ is a PID and a UFD.