Polynomial Rings that are UFDs

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Proposition (Gauss's Lemma)

Let R be a UFD with field of fractions F and let $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$ then $p(x)$ is reducible in $R[x]$.More precisely, if $p(x) = A(x)B(x)$ for some non-constant $A, B \in F[x]$, then there are $0 \neq r, s \in F$ such that $a(x) = rA(x)$ and $b(x) = sB(x)$ lie in $R[x]$ with $p(x) = a(x)b(x)$.

Corollary

Let R be a UFD with field of fractions F and let $p(x) \in R[x]$. Suppose also that the gcd of the coefficients of $p(x)$ is 1. Then $p(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. In particular, if $p(x)$ is monic and irreducible in $R[x]$ then $p(x)$ is irreducible in $F[x]$.

Theorem

R is a UFD if and only if $R[x]$ is a UFD.

Corollary

If R is a UFD, then a polynomial ring in an arbitrary number of variables with coefficients in R is also a UFD.