Irreducibility Criterion

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Proposition

Let F be a field and let $p \in F[x]$. Then $p(x)$ has a factor of degree one if and only if $p(x$ has a root in F.

Proposition

A polynomial of degree 2 or 3 over a field F is reducible if and only if it has a root in F.

Proposition

Suppose that $p(x) = \sum_{k=0}^{n} p_n x^n \in \mathbb{Q}[x]$. If $r/s \in \mathbb{Q}$ is in lowest terms and is a root of p then r divides p_0 and s divides p_n.

Proposition

Suppose that R is an ID and $I \trianglelefteq R$. Let $p \in R[x]$ be nonconstant and monic. If the image of p in $(R/I)[x]$ cannot be factored into to polynomials of lower degree, then $(R/I)[x]$ then $p(x)$ is irreducible in $R[x]$.

Proposition (Eisenstin's Criterion)

Let P be a prime ideal of an integral domain R and let $f(x) = x^n + \sum_{n=0}^{n-1} f_n x^n \in R[x]$ $(n \geq 1)$. Suppose $f_{n-1}, \ldots, f_1, f_0 \in P$, and $f_0 \notin P^2$. Then $f(x)$ is irreducible in $R[x]$.

Corollary (Eisenstein's Criterion for $\mathbb{Z}[x]$)

Let $p \in \mathbb{Z}$ be prime and let $f(x) = x^n + \sum_{n=0}^{n-1} f_n x^n \in \mathbb{Z}[x]$ with $n \geq 1$. Suppose that $p | f_i$ for $1 \leq i \leq (n-1)$ and that $p^2 \nmid f_0$. Then $f(x)$ is irreducible in $\mathbb{Z}[x]$ and in $\mathbb{Q}[x]$.