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Average Frobenius distribution for elliptic curves defined over finite Galois extensions of the rationals

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Abstract

Let K be a fixed number field, assumed to be Galois over $\mathbb{Q}$. Let r and f be fixed integers with f positive. Given an elliptic curve E, defined over K, we consider the problem of counting the number of degree f prime ideals of K with trace of Frobenius equal to r. Except in the case $f = 2$, we show that 'on average,' the number of such prime ideals with norm less than or equal to x satisfies an asymptotic identity that is in accordance with standard heuristics. This work is related to the classical Lang–Trotter conjecture and extends the work of several authors.

1. *Introduction*

We begin by reviewing the classical case. Let E be an elliptic curve defined over the rational field $\mathbb{Q}$. For a prime p where E has good reduction, we let $a_p(E)$ denote the trace of the Frobenius morphism. Then $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$, and it was shown by Hasse that $|a_p(E)| \leqslant 2\sqrt{p}$. See [**20**, p. 131]. Given a fixed elliptic curve E and a fixed integer r, the prime counting function

$$\pi_E^r(x) := \#\{p \leqslant x : a_p(E) = r\} \tag{1}$$

has received a great deal of attention. Deuring [**7**] showed that if E admits complex multiplication, then half the primes are of *supersingular* reduction, i.e., $a_p(E) = 0$. In addition, the distribution of the non-supersingular primes was explained by Hecke [**9**, **10**] for elliptic curves with complex multiplication. For the remaining cases, we have the following conjecture of Lang and Trotter [**16**].

CONJECTURE 1 (Lang–Trotter). *Let E be a fixed elliptic curve defined over $\mathbb{Q}$, and let r be a fixed integer. In the case that E has complex multiplication, also assume that $r \neq 0$.*

There exists a constant $C_{E,r}$ such that

$$\pi_E^r(x) \sim C_{E,r}\frac{\sqrt{x}}{\log x} \tag{2}$$

as $x \to \infty$. The constant $C_{E,r}$ may be zero, in which case the asymptotic is interpreted to mean that there are only finitely many such primes.

The theme of studying this conjecture "on average" was initiated by Fouvry and Murty in [**8**] who considered the case when $r = 0$. The density of complex multiplication curves is so little that they do not affect the asymptotic. Their work was generalized by David and Pappalardi [**5**] who considered the remaining cases. This was later improved by Baier [**1**] who showed that the result also holds over a "shorter average." Finer averages have also been considered. In [**12**], the first author considered the problem when the average was restricted to curves admitting a rational 3-torsion point. Averages over families of elliptic curves with various prescribed torsion structures were considered in [**2**].

We now turn to the number field case. Suppose that K is a number field and E is an elliptic curve defined over K. Given a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ where E has good reduction, we define the trace of Frobenius $a_{\mathfrak{p}}(E)$ as before. In particular, we have $a_{\mathfrak{p}}(E) = \mathrm{N}\mathfrak{p} + 1 - \#E(\mathcal{O}_K/\mathfrak{p})$ and $|a_{\mathfrak{p}}(E)| \leqslant 2\sqrt{\mathrm{N}\mathfrak{p}} = 2p^{f/2}$. Here, $\mathrm{N}\mathfrak{p} := \#(\mathcal{O}_K/\mathfrak{p}) = p^f$ is the norm of $\mathfrak{p}$, p is the unique rational prime lying below $\mathfrak{p}$, and $f = \deg \mathfrak{p}$ is the absolute degree of $\mathfrak{p}$. For a fixed elliptic curve E and fixed integers r and f, we define the prime counting function

$$\pi_E^{r,f}(x) := \#\{\mathrm{N}\mathfrak{p} \leqslant x : a_{\mathfrak{p}}(E) = r \text{ and } \deg \mathfrak{p} = f\}. \tag{3}$$

For elliptic curves defined over a number field K, the heuristics of Lang and Trotter [**16**] suggest the following more refined conjecture. See also [**6**].

CONJECTURE 2 (Lang–Trotter for number fields). *Let E be a fixed elliptic curve defined over K, and let r be a fixed integer. In the case that E has complex multiplication, also assume that $r \neq 0$. Let f be a positive integer. There exists a constant $\mathfrak{C}_{E,r,f}$ such that*

$$\pi_E^{r,f}(x) \sim \mathfrak{C}_{E,r,f}\begin{cases} \frac{\sqrt{x}}{\log x} & \text{if } f = 1, \\ \log\log x & \text{if } f = 2, \\ 1 & \text{if } f \geqslant 3, \end{cases} \tag{4}$$

as $x \to \infty$. The constant $\mathfrak{C}_{E,r,f}$ may be zero, in which case the asymptotic is interpreted to mean that there are only finitely many such primes.

Remark. For a fixed $f \geqslant 3$, we interpret the conjecture to say that there are only finitely many such primes. In this case, the constant $\mathfrak{C}_{E,r,f}$ would necessarily be a nonnegative integer.

This conjecture too has been studied on average. David and Pappalardi [**6**] considered the case when $K = \mathbb{Q}(i)$ and $f = 2$. A recent paper of Calkin, Faulkner, King, Penniston and the first author [**3**] extended their work to the setting of an arbitrary number field K assumed to be Abelian over $\mathbb{Q}$. In fact, the authors of [**3**] considered any positive integer f and obtained asymptotics in accordance with the conjecture.

The purpose of this paper is to improve the work of [**3**] in two ways. In the first place, we will relax the assumption that the number field K is Abelian over $\mathbb{Q}$ provided that $f \neq 2$.

Instead, we will assume that K is Galois over $\mathbb{Q}$. The case $f = 2$ remains somewhat elusive. However, the authors are currently pursuing this case. The second improvement is that we will consider a "more general average" which will allow us to show that Conjecture 2 still holds on average when averaging over a "smaller" set of elliptic curves.

2. *Statement of results*

For the remainder of the paper, we will assume that K is a fixed number field. In addition, we assume that the extension $K/\mathbb{Q}$ is Galois. We denote the degree of the extension by $n_K := [K : \mathbb{Q}]$. Recall that $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n_K and let $\mathcal{B} = \{\gamma_j\}_{j=1}^{n_K}$ be a fixed integral basis for $\mathcal{O}_K$. We denote the coordinate map for the basis $\mathcal{B}$ by

$$[\,\cdot\,]_{\mathcal{B}} : \mathcal{O}_K \xrightarrow{\sim} \bigoplus_{j=1}^{n_K} \mathbb{Z} = \mathbb{Z}^{n_K}.$$

Given two vectors $\mathbf{a},\mathbf{b} \in \mathbb{Z}^{n_K}$, if each entry of $\mathbf{a}$ is less than or equal to the corresponding entry of $\mathbf{b}$, then we write $\mathbf{a} \leqslant \mathbf{b}$. If $\mathbf{b} \geqslant \mathbf{0}$, then we define a "box" of algebraic integers by

$$\mathcal{B}(\mathbf{a}, \mathbf{b}) := \{\alpha \in \mathcal{O}_K : \mathbf{a} - \mathbf{b} \leqslant [\alpha]_{\mathcal{B}} \leqslant \mathbf{a} + \mathbf{b}\}. \tag{5}$$

For two algebraic integers $\alpha, \beta \in \mathcal{O}_K$, we write $E_{\alpha,\beta}$ for the elliptic curve given by the model

$$E_{\alpha,\beta} : Y^2 = X^3 + \alpha X + \beta.$$

Then for appropriate vectors, we define a "box" of elliptic curves by

$$\mathscr{B} := \mathscr{B}(\mathbf{a}_1, \mathbf{b}_1; \mathbf{a}_2, \mathbf{b}_2) = \{E_{\alpha,\beta} : \alpha \in \mathcal{B}(\mathbf{a}_1, \mathbf{b}_1) \text{ and } \beta \in \mathcal{B}(\mathbf{a}_2, \mathbf{b}_2)\}. \tag{6}$$

To be more precise, this box should be thought of as a box of equations or models since the same elliptic curve may appear multiple times in $\mathscr{B}$. For $i = 1, 2$, let $b_{i,j}$ denote the jth entry of $\mathbf{b}_i$. Associated to box $\mathscr{B}$, we define the quantities

$$\mathscr{V}(\mathscr{B}) = 2^{2n_K} \prod_{j=1}^{n_K} b_{1,j} * b_{2,j}, \tag{7}$$

$$\mathscr{V}_1(\mathscr{B}) = 2^{n_K} \prod_{j=1}^{n_K} b_{1,j}, \tag{8}$$

$$\mathscr{V}_2(\mathscr{B}) = 2^{n_K} \prod_{j=1}^{n_K} b_{2,j}, \tag{9}$$

$$\mathscr{V}_{\min}(\mathscr{B}) = 2 \min_{1 \leqslant j \leqslant n_K} \{b_{1,j}, b_{2,j}\}, \tag{10}$$

which give a description of the size of this box. In particular,

$$\#\mathscr{B} = \mathscr{V}(\mathscr{B}) + O(\mathscr{V}(\mathscr{B})/\mathscr{V}_{\min}(\mathscr{B})).$$

Recall that

$$\pi_{1/2}(x) := \int_2^x \frac{dt}{2\sqrt{t}\log t} \sim \frac{\sqrt{x}}{\log x}. \tag{11}$$

We are now ready to state the main results of this paper.

THEOREM 1. *Let r be a fixed integer. Then, for any $\eta > 0$,*

$$\frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \pi_E^{r,1}(x) = \mathfrak{C}_{K,r,1}\pi_{1/2}(x) + \mathcal{E}(x; \mathscr{B}),$$

where

$$\mathcal{E}(x; \mathscr{B}) \ll \frac{\sqrt{x}}{(\log x)^{1+\eta}} + \frac{\sqrt{x}/\log x}{\mathscr{V}_{\min}(\mathscr{B})} + \left(\frac{1}{\mathscr{V}_1(\mathscr{B})} + \frac{1}{\mathscr{V}_2(\mathscr{B})}\right)(x \log x)^{n_K} + \frac{(x \log x)^{2n_K}}{\mathscr{V}(\mathscr{B})},$$

and $\mathfrak{C}_{K,r,1}$ is the constant defined by the absolutely convergent sum (12) *in Section 3.*

As an immediate corollary of Theorem 1, we have the following:

COROLLARY 1. *Let $\eta > 0$, and let r be a fixed integer. Then*

$$\frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \pi_E^{r,1}(x) \sim \mathfrak{C}_{K,r,1}\pi_{1/2}(x),$$

provided that the box $\mathscr{B}$ satisfies the growth conditions:

$$\begin{aligned} \mathscr{V}(\mathscr{B}) &\gg x^{2n_K-1/2}(\log x)^{2n_K+1+\eta}, \\ \mathscr{V}_1(\mathscr{B}), \mathscr{V}_2(\mathscr{B}) &\gg x^{n_K-1/2}(\log x)^{n_K+1+\eta}, \\ \mathscr{V}_{min}(\mathscr{B}) &\gg (\log x)^{\eta}. \end{aligned}$$

Remark. The "growth rate" of the box $\mathscr{B}$ is much smaller than that of the corresponding box in [**3**].

We also consider the mean square error or how much the function $\pi_E^{r,1}(x)$ varies from the average $\mathfrak{C}_{K,r,1}\pi_{1/2}(x)$.

THEOREM 2. *Let $\eta > 0$. Then*

$$\frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \left|\pi_E^{r,1}(x) - \mathfrak{C}_{K,r,1}\pi_{1/2}(x)\right|^2 \ll \frac{x}{(\log x)^{2+\eta}},$$

provided that the box $\mathscr{B}$ satisfies the growth conditions:

$$\begin{aligned} \mathscr{V}(\mathscr{B}) &\gg x^{4n_K-1}(\log x)^{4n_K+2+\eta}, \\ \mathscr{V}_1(\mathscr{B}), \mathscr{V}_2(\mathscr{B}) &\gg x^{2n_K-1}(\log x)^{2n_K+2+\eta}, \\ \mathscr{V}_{min}(\mathscr{B}) &\gg (\log x)^{\eta}. \end{aligned}$$

An application of the Turán normal order method (see [**4**, chapter 3]) supplies us with the following corollary.

COROLLARY 2. *Let $\delta, \eta > 0$ be fixed with $\delta > 2\eta$. If $\mathscr{B}$ satisfies the conditions of Theorem 2, then for all $E \in \mathscr{B}$ with at most $O(\frac{\mathscr{V}(\mathscr{B})}{(\log x)^{\delta-2\eta}})$ exceptions, we have*

$$\left|\pi_E^{r,1}(x) - \mathfrak{C}_{K,r,1}\pi_{1/2}(x)\right| < \frac{\sqrt{x}}{(\log x)^{1+\eta}}.$$

Remark. Care should be taken with the interpretation of Corollary 2. It would be easy to draw the conclusion that the average order constant $\mathfrak{C}_{K,r,1}$ is the correct constant $\mathfrak{C}_{E,r,1}$ for most elliptic curves E defined over K. Although it is possible for this to happen for a given

choice of K and r, this is not implied by Corollary 2. In fact, Corollary 2 does not even imply that there is one elliptic curve E for which $\mathfrak{C}_{K,r,1} = \mathfrak{C}_{E,r,1}$. The key fact to remember when interpreting Corollary 2 is that the curves appearing in the box $\mathscr{B}$ depend on x. Therefore, as x changes, so might the exceptional set. In fact, it is possible that for a given value of r, every elliptic curve defined over K "eventually" enters the exceptional set.

For $f \geqslant 3$, we also have the following average order result for $\pi_E^{r,f}(x)$.

THEOREM 3. *Let r be a fixed integer. If $f \geqslant 3$, then*

$$\lim_{x\to\infty} \frac{1}{\#\mathscr{B}} \sum_{E\in\mathscr{B}} \pi_E^{r,f}(x) < \infty,$$

provided that $\mathscr{V}_{min}(\mathscr{B}) \gg x^{1/f}$.

Remark. The authors of [**3**] state a version of this result under the assumption that $K/\mathbb{Q}$ is Abelian. Although they only state their result for Abelian extensions, it turns out that their methods are sufficient to prove Theorem 3 under the relaxed assumption that $K/\mathbb{Q}$ is Galois. Therefore, we will omit the proof of Theorem 3 and concentrate on the case $f = 1$ in this paper.

Remark. Unfortunately, using the techniques that we present here, the case $f = 2$ is not so easily generalized to the setting of an arbitrary finite, normal extension of $\mathbb{Q}$. See the discussion in Section 8 preceding the proof of Proposition 1. However, the authors of this paper do believe it should be possible to make some progress towards such a result and are currently pursuing it for a certain class of number fields possessing a non-Abelian Galois group over $\mathbb{Q}$.

3. *The average order constant*

In this section, we give a precise description of the constant $\mathfrak{C}_{K,r,1}$, first as an infinite sum and then as an infinite product over primes. This requires a considerable amount of additional notation.

Recall that $G = \mathrm{G}al(K/\mathbb{Q})$. Let $[G, G]$ denote the commutator subgroup of G, and let $\mathcal{A}$ be the fixed field of $[G, G]$. Then $\mathcal{A}/\mathbb{Q}$ is an Abelian extension of finite degree, which we denote by $n_{\mathcal{A}} = [\mathcal{A} : \mathbb{Q}]$. By the Kronecker–Weber Theorem [**15**, p. 210], it follows that there is a smallest positive integer m_K so that $\mathcal{A} \subseteq \mathbb{Q}(\zeta_{m_K})$. Here, ζ_{m_K} is a primitive m_K-th root of unity. It is well-known that $\mathrm{G}al(\mathbb{Q}(\zeta_{m_K})/\mathbb{Q}) \cong (\mathbb{Z}/m_K\mathbb{Z})^*$ with a natural choice of isomorphism. See [**23**, p. 11] for example. Let G_{m_K} denote the subgroup of $(\mathbb{Z}/m_K\mathbb{Z})^*$ corresponding to $\mathrm{G}al(\mathbb{Q}(\zeta_{m_K})/\mathcal{A})$ under this isomorphism.

The constant $\mathfrak{C}_{K,r,1}$ is given by

$$\mathfrak{C}_{K,r,1} := \frac{2n_{\mathcal{A}}}{\pi} \left(\sum_{b\in G_{m_K}} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_k^{r,b,m_K}(n)}{nk\varphi([m_K, nk^2])} \right), \tag{12}$$

where

$$c_k^{r,b,m_K}(n) := \sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z}) \\ a\equiv 0,1 \pmod 4 \\ (r^2-ak^2,4nk^2)=4 \\ 4b\equiv r^2-ak^2 \pmod{(4m_K,4nk^2)}}} \left(\frac{a}{n}\right). \tag{13}$$

The fact that the double infinite sum in (12) is absolutely convergent follows from the proof of Proposition 1. See Section 4. However, since the proof of Proposition 1 is similar to the proof of [**6**, lemma 2·2], we give only a sketch of the proof, highlighting the differences. See [**6**, pp. 193–199] for more detail.

We also have a description of the constant $\mathfrak{C}_{K,r,1}$ as a product. However, this requires the introduction of some more notation. For $b \in G_{m_K}$, let $\Delta^{r,b} := r^2 - 4b$, and define the following sets of rational primes

$$\mathfrak{Q}^{<}_{r,b,m_K} := \{\ell > 2 : \ell | m_K, \ell \nmid r, \text{ and } \mathrm{ord}_\ell(\Delta^{r,b}) < \mathrm{ord}_\ell(m_K)\}, \tag{14}$$

$$\mathfrak{Q}^{\geqslant}_{r,b,m_K} := \{\ell > 2 : \ell | m_K, \ell \nmid r, \text{ and } \mathrm{ord}_\ell(\Delta^{r,b}) \geqslant \mathrm{ord}_\ell(m_K)\}. \tag{15}$$

In addition, let

$$\Gamma_\ell := \begin{cases} \left(\frac{\Delta^{r,b}/\ell^{\mathrm{ord}_\ell(\Delta^{r,b})}}{\ell}\right) & \text{if } \mathrm{ord}_\ell(\Delta^{r,b}) \text{ is even, positive, and finite,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{F}_2(r,b,m_K) := \begin{cases} 2/3 & \text{if } 2 \nmid r; \\ 4/3 & \text{if } 2|r, 4 \nmid m_K; \\ 2 - \frac{2}{3\cdot 2^{\lfloor \mathrm{ord}_2(m_K)/2 \rfloor}} & \text{if } r \equiv 2 \pmod 4, 2 \leqslant \mathrm{ord}_2(m_K) \leqslant \mathrm{ord}_2(\Delta^{r,b}) - 2; \\ 2 - \frac{4}{3\cdot 2^{\frac{\mathrm{ord}_2(m_K)-1}{2}}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) = \mathrm{ord}_2(\Delta^{r,b}) - 1, \\ & \quad 2|\mathrm{ord}_2(\Delta^{r,b}); \\ 2 - \frac{2}{2^{\mathrm{ord}_2(m_K)/2}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) = \mathrm{ord}_2(\Delta^{r,b}) - 1, \\ & \quad 2 \nmid \mathrm{ord}_2(\Delta^{r,b}); \\ 2 - \frac{2}{3\cdot 2^{\mathrm{ord}_2(m_K)/2}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) = \mathrm{ord}_2(\Delta^{r,b}), \\ & \quad 2|\mathrm{ord}_2(\Delta^{r,b}), \frac{\Delta^{r,b}}{2^{\mathrm{ord}_2(\Delta^{r,b})}} \equiv 1 \pmod 4; \\ 2 - \frac{2}{2^{\lfloor \mathrm{ord}_2(m_K)/2 \rfloor}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) = \mathrm{ord}_2(\Delta^{r,b}), \\ & \quad \left[2 \nmid \mathrm{ord}_2(\Delta^{r,b}) \textbf{ OR } \frac{\Delta^{r,b}}{2^{\mathrm{ord}_2(\Delta^{r,b})}} \equiv 3 \pmod 4\right]; \\ 2 & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) > \mathrm{ord}_2(\Delta^{r,b}), \\ & \quad 2|\mathrm{ord}_2(\Delta^{r,b}), \frac{\Delta^{r,b}}{2^{\mathrm{ord}_2(\Delta^{r,b})}} \equiv 1 \pmod 8; \\ 2 - \frac{4}{3\cdot 2^{\mathrm{ord}_2(\Delta^{r,b})/2}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) > \mathrm{ord}_2(\Delta^{r,b}), \\ & \quad 2|\mathrm{ord}_2(\Delta^{r,b}), \frac{\Delta^{r,b}}{2^{\mathrm{ord}_2(\Delta^{r,b})}} \equiv 5 \pmod 8; \\ 2 - \frac{2}{2^{\mathrm{ord}_2(\Delta^{r,b})/2}} & \text{if } r \equiv 2 \pmod 4, \mathrm{ord}_2(m_K) > \mathrm{ord}_2(\Delta^{r,b}), \\ & \quad \left[2 \nmid \mathrm{ord}_2(\Delta^{r,b}) \textbf{ OR } \frac{\Delta^{r,b}}{2^{\mathrm{ord}_2(\Delta^{r,b})}} \equiv 3 \pmod 8\right]; \\ \frac{5}{3} & \text{if } r \equiv 0 \pmod 4, \mathrm{ord}_2(m_K) = 2, b \equiv 3 \pmod 4; \\ 2 & \text{if } r \equiv 0 \pmod 4, 8|m_K, b \equiv 3 \pmod 4, \\ & \quad \frac{\Delta^{r,b}}{4} \equiv 1 \pmod 8; \\ \frac{4}{3} & \text{if } r \equiv 0 \pmod 4, 8|m_K, b \equiv 3 \pmod 4, \\ & \quad \frac{\Delta^{r,b}}{4} \equiv 5 \pmod 8; \\ 1 & \text{if } r \equiv 0 \pmod 4, 4|m_K, b \equiv 1 \pmod 4. \end{cases}$$

Finally, let $\mathcal{F}(r, b, m_K)$ denote the following finite product over the primes dividing m_K:

$$\mathcal{F}_2(r, b, m_K) \prod_{\substack{\ell \neq 2 \\ \ell | m_K \\ \ell | r}} \frac{\ell\left(\ell + \left(\frac{-b}{\ell}\right)\right)}{\ell^2 - 1} \prod_{\ell \in \mathfrak{Q}^{\geqslant}_{r,b,m_K}} \left(\frac{\ell^{\left\lfloor \frac{\mathrm{ord}_\ell(m_K)+1}{2} \right\rfloor} - 1}{\ell^{\left\lfloor \frac{\mathrm{ord}_\ell(m_K)-1}{2} \right\rfloor}(\ell - 1)} + \frac{\ell^{\mathrm{ord}_\ell(m_K)+2}}{\ell^{3\left\lfloor \frac{\mathrm{ord}_\ell(m_K)+1}{2} \right\rfloor}(\ell^2 - 1)} \right)$$
$$\cdot \prod_{\ell \in \mathfrak{Q}^{<}_{r,b,m_K}} \left(1 + \frac{\ell\left(\frac{\Delta^{r,b}}{\ell}\right) + \left(\frac{\Delta^{r,b}}{\ell}\right)^2 + \frac{\ell\Gamma_\ell + \ell^2\Gamma_\ell^2}{\ell^{\mathrm{ord}_\ell(\Delta^{r,b})/2}}}{\ell^2 - 1} + \frac{\Gamma_\ell^2\left(\ell^{\left\lfloor \frac{\mathrm{ord}_\ell(\Delta^{r,b})-1}{2} \right\rfloor} - 1\right)}{\ell^{\left\lfloor \frac{\mathrm{ord}_\ell(\Delta^{r,b})-1}{2} \right\rfloor}(\ell - 1)} \right). \tag{16}$$

THEOREM 4. *As an infinite product over primes, we may write*

$$\mathfrak{C}_{K,r,1} = \left(\frac{2n_{\mathcal{A}}}{\pi\varphi(m_K)} \prod_{\substack{\ell \neq 2 \\ \ell \nmid m_K \\ \ell \nmid r}} \frac{\ell(\ell^2 - \ell - 1)}{(\ell + 1)(\ell - 1)^2} \prod_{\substack{\ell \neq 2 \\ \ell \nmid m_K \\ \ell | r}} \frac{\ell^2}{\ell^2 - 1} \right) \sum_{b \in G_{m_K}} \mathcal{F}(r, b, m_K).$$

Remark. In the case that $K/\mathbb{Q}$ is an Abelian extension, $K = \mathcal{A}$ and the constant $\mathfrak{C}_{K,r,1}$ agrees with the average order constant of [**3**] (stated only for odd r).

Proof. See [**13**, theorem 1·1, proposition 2·1] of [**13**] and compare with (12).

4. *Intermediate results*

4·1. *Counting curves*

For a fixed prime ideal $\mathfrak{p}$ and a fixed elliptic curve E defined over the finite field $\mathcal{O}_K/\mathfrak{p}$, we will need to count the number of models in the box $\mathscr{B}$ which are isomorphic to E modulo $\mathfrak{p}$. Now, if we assume that $\mathfrak{p} \nmid 6$, then any elliptic curve defined over $\mathcal{O}_K/\mathfrak{p}$ may be realized using a model of the form

$$E_{a,b} : Y^2 = X^3 + aX + b \text{ with } a, b \in \mathcal{O}_K/\mathfrak{p}. \tag{17}$$

For an elliptic curve E over K and a prime $\mathfrak{p}$ of good reduction, let $E^{\mathfrak{p}}$ denote the reduction of E modulo $\mathfrak{p}$. In order to carry out the proof of Theorem 1, we will need an estimate on the size of

$$\mathscr{B}(E_{a,b}, \mathfrak{p}) := \{E \in \mathscr{B} : E^{\mathfrak{p}} \cong E_{a,b}\}. \tag{18}$$

Remark. It is important to note that here we are counting equations (or models) in $\mathscr{B}$ whose reduction modulo $\mathfrak{p}$ are in the same isomorphism class as $E_{a,b}$.

LEMMA 1. *Recall that $\mathcal{B} = \{\gamma_j\}_{j=1}^{n_K}$ is our fixed integral basis for $\mathcal{O}_K$. Let $\mathfrak{p}$ be a degree 1 prime of K such that $\mathfrak{p} \nmid 6\prod_{j=1}^{n_K}\gamma_j$. Let p denote the unique rational prime lying below $\mathfrak{p}$, and let $E_{a,b}$ be the fixed elliptic curve defined over $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_p$ via the equation $E_{a,b}$: $Y^2 = X^3 + aX + b$. Then*

$$\#\mathscr{B}(E_{a,b}, \mathfrak{p}) = \frac{p-1}{p^2 \#Aut_p(E_{a,b})}\mathscr{V}(\mathscr{B}) + O\left(\frac{\mathscr{V}(\mathscr{B})}{p\mathscr{V}_{min}(\mathscr{B})} + (\mathscr{V}_1(\mathscr{B}) + \mathscr{V}_2(\mathscr{B}))p^{n_K - 3/2}(\log p)^{n_K} \right) + O\left(p^{2n_K - 3/2}(\log p)^{2n_K}\right),$$

where

$$\#Aut_p(E_{a,b}) = \begin{cases} 2 & \text{if } ab \neq 0, \\ (4,\, p-1) & \text{if } a \neq 0 \text{ and } b = 0, \\ (6,\, p-1) & \text{if } a = 0 \text{ and } b \neq 0. \end{cases}$$

We delay the proof of Lemma 1 until Section 7. In addition, if $\mathfrak{p}$ and $\mathfrak{p}'$ do not lie above the same rational prime, we will also need an estimate on the size of

$$\mathscr{B}(E_{a,b}, \mathfrak{p}; E_{a',b'}, \mathfrak{p}') := \{E \in \mathscr{B} : E^{\mathfrak{p}} \cong E_{a,b} \text{ and } E^{\mathfrak{p}'} \cong E_{a',b'}\} \tag{19}$$

in order to carry out the proof of Theorem 2.

LEMMA 2. *With the same notation and assumptions as in Lemma* 1*, assume further that* $\mathfrak{p}'$ *is a prime of* K *satisfying the same conditions as* $\mathfrak{p}$ *except that* $\mathfrak{p}'$ *lies over the rational prime* p' *and* $p' \neq p$*. That is,* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *do not lie over the same prime. Also, let* $E_{a',b'}$ *be the fixed elliptic curve defined over* $\mathcal{O}_K/\mathfrak{p}' \cong \mathbb{F}_{p'}$ *via the equation* $E_{a',b'} : Y^2 = X^3 + a'X + b'$*. Then*

$$\begin{aligned} \#\mathscr{B}(E_{a,b}, \mathfrak{p}; E_{a',b'}, \mathfrak{p}') = {} & \frac{(p-1)(p'-1)}{(pp')^2 \#Aut_p(E_{a,b}) \#Aut_{p'}(E_{a',b'})} \mathscr{V}(\mathscr{B}) + O\left(\frac{\mathscr{V}(\mathscr{B})}{pp' \mathscr{V}_{min}(\mathscr{B})}\right) \\ & + O((pp')^{2n_K - 3/2} (\log pp')^{2n_K} \\ & + (\mathscr{V}_1(\mathscr{B}) + \mathscr{V}_2(\mathscr{B}))(pp')^{n_K - 3/2})(\log pp')^{n_K}). \end{aligned}$$

As the proof of Lemma 2 is similar to the proof of Lemma 1, we omit it.

4·2. *A weighted average of special values of Dirichlet L-functions*

Recall that for a Dirichlet character χ and for $\Re(s)$ sufficiently large, the Dirichlet L-function associated to χ is defined by

$$L(s, \chi) := \sum_{n \geqslant 1} \frac{\chi(n)}{n^s}.$$

Also, recall that when χ is not trivial, this series converges at $s = 1$. For an integer d, we let χ_d denote the Kronecker symbol $\left(\frac{d}{\cdot}\right)$.

Let $B(r) := \max\{5, r^2/4\}$, and let $\mathscr{P}_r$ denote the set of prime ideals $\mathfrak{p}$ satisfying:

(i) $B(r) < \mathrm{N}\mathfrak{p}$;

(ii) $\mathfrak{p}$ lies over some rational prime p which splits completely in K;

(iii) $\mathfrak{p} \nmid \prod_{j=1}^{n_K} \gamma_j$;

(iv) $\mathfrak{p}$ does not ramify in $K(\zeta_{m_K})$.

We also define the corresponding set of "downstairs" primes. That is, we let $\mathcal{P}_r$ denote the set of rational primes p lying below some prime $\mathfrak{p} \in \mathscr{P}_r$. In addition, we will require the "truncated" sets $\mathscr{P}_r(x) := \{\mathfrak{p} \in \mathscr{P}_r : \mathrm{N}\mathfrak{p} \in (1, x]\}$ and $\mathcal{P}_r(x) := \mathcal{P}_r \cap (1, x]$.

For a prime p and a positive integer k, let $d_k(p) := (r^2 - 4p)/k^2$ if k^2 divides $r^2 - 4p$. Finally, define the set

$$\mathcal{S}_k(x; r) := \{p \in \mathcal{P}_r(x) : k^2 | (4p - r^2), d_k(p) \equiv 0, 1 \pmod 4\}. \tag{20}$$

PROPOSITION 1. *Let*

$$A_1(x;r) := n_K \sum_{k \leqslant 2\sqrt{x}} \frac{1}{k} \sum_{p \in \mathcal{S}_k(x;r)} L(1, \chi_{d_k(p)}) \log p.$$

Then the double infinite sum defining $\mathfrak{C}_{K,r,1}$ *in* (12) *is absolutely convergent; and for any* $\eta > 0$,

$$A_1(x;r) = \frac{\pi}{2}\mathfrak{C}_{K,r,1}x + O\left(\frac{x}{(\log x)^{\eta}}\right).$$

A sketch of the proof of Proposition 1 is given in Section 8. For the omitted details, we refer the reader to [**6**, pp. 193–199] since it is similar.

5. *The average order*

We now use the results of Section 4 to compute the average order of $\pi_E^{r,1}(x)$. That is, we prove Theorem 1. We compute the average order by first converting it into a weighted sum of class numbers. Given a (not necessarily fundamental) discriminant $D < 0$, we define the *Hurwitz-Kronecker class number* of discriminant D by

$$H(D) := 2 \sum_{\substack{k^2 \mid D \\ \frac{D}{k^2} \equiv 0,1 \pmod 4}} \frac{h(D/k^2)}{w(D/k^2)}, \tag{21}$$

where $h(d)$ denotes the class number of the unique imaginary quadratic order of discriminant d and $w(d)$ denotes the order of its unit group.

The following result of Deuring is the key to counting elliptic curves over a finite field. See [**7**] or [**17**, p. 654].

THEOREM 5 (Deuring). *Let p be prime greater than* 3 *and r an integer satisfying* $r^2 - 4p < 0$. *Then*

$$\sum_{\substack{\tilde{E}/\mathbb{F}_p \\ \#\tilde{E}(\mathbb{F}_p) = p+1-r}} \frac{1}{\#Aut(\tilde{E})} = \frac{1}{2}H(r^2 - 4p),$$

where the sum on the left is over the $\mathbb{F}_p$-isomorphism classes of elliptic curves having exactly $p + 1 - r$ points and $\#Aut(\tilde{E})$ denotes the size of the automorphism group of any representative of the class $\tilde{E}$.

Remark. It is important to note that our definition of $H(D)$ is defined as a weighted sum of ordinary class numbers $h(d)$. Thus, our statement of Deuring's Theorem looks more like that given in [**17**, p. 654] as opposed to that given in [**19**, theorem 4·6]. However, our definition is exactly twice as big as the definition used in [**17**] and the statement of Deuring's Theorem is adjusted accordingly.

PROPOSITION 2. *If $\mathcal{P}_r(x)$ is the set of primes defined in Section 4·2, then*

$$\frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \pi_E^{r,1}(x) = \frac{n_K}{2} \sum_{p \in \mathcal{P}_r(x)} \frac{H(r^2 - 4p)}{p} + O(\mathcal{E}_0(x;\mathscr{B})),$$

where

$$\mathcal{E}_0(x;\mathscr{B}) := 1 + \frac{\sqrt{x}/\log x}{\mathscr{V}_{min}(\mathscr{B})} + \left(\frac{1}{\mathscr{V}_1(\mathscr{B})} + \frac{1}{\mathscr{V}_2(\mathscr{B})}\right)(x\log x)^{n_K} + \frac{(x\log x)^{2n_K}}{\mathscr{V}(\mathscr{B})}.$$

Proof. Since $\mathscr{P}_r$ contains all but finitely many degree 1 primes of K, we have

$$\begin{aligned}\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\pi_E^{r,1}(x) &= \frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathfrak{p}\in\mathscr{P}_r(x)\\ a_{\mathfrak{p}}(E)=r}} 1 + O(1) = \frac{1}{\#\mathscr{B}}\sum_{\mathfrak{p}\in\mathscr{P}_r(x)}\sum_{\substack{E\in\mathscr{B}\\ a_{\mathfrak{p}}(E)=r}} 1 + O(1)\\ &= \frac{1}{\#\mathscr{B}}\sum_{\mathfrak{p}\in\mathscr{P}_r(x)}\sum_{\substack{\tilde{E}/\mathbb{F}_p\\ \#\tilde{E}(\mathbb{F}_p)=p+1-r}} \#\mathscr{B}(\tilde{E},\mathfrak{p}) + O\left(\frac{1}{\#\mathscr{B}}\sum_{\mathfrak{p}\in\mathscr{P}_r(x)}\sum_{\substack{E\in\mathscr{B}\\ E_{\mathfrak{p}}\text{ sing.}}} 1\right),\end{aligned} \tag{22}$$

where $p = \mathrm{N}\mathfrak{p}$ and the inner sum of the final big-O term is over the $E \in \mathscr{B}$ whose reductions are singular modulo $\mathfrak{p}$. Using character sums in a manner similar to the proof of Lemma 1 (see Section 7), we obtain a bound on the big-O term which is smaller than $\mathcal{E}_0(x;\mathscr{B})$. Therefore, we will concentrate on the main term of (22).

By Lemma 1, we see that $\#\mathscr{B}(\tilde{E},\mathfrak{p})$ depends only on the rational prime p lying below $\mathfrak{p}$. Since there are exactly n_K degree 1 primes $\mathfrak{p}$ lying above the same rational prime p, we obtain

$$\frac{1}{\#\mathscr{B}}\sum_{\mathfrak{p}\in\mathscr{P}_r(x)}\sum_{\substack{\tilde{E}/\mathbb{F}_p\\ \#\tilde{E}(\mathbb{F}_p)=p+1-r}} \#\mathscr{B}(\tilde{E},\mathfrak{p}) = \frac{n_K}{\#\mathscr{B}}\sum_{p\in\mathcal{P}_r(x)}\sum_{\substack{\tilde{E}/\mathbb{F}_p\\ \#\tilde{E}(\mathbb{F}_p)=p+1-r}} \#\mathscr{B}(\tilde{E},\mathfrak{p}).$$

Finally, using Lemma 1 to estimate $\#\mathscr{B}(\tilde{E},\mathfrak{p})$ and Theorem 5 to count $\mathbb{F}_p$-isomorphism classes of elliptic curves with exactly $p+1-r$ points, we obtain

$$\frac{1}{\#\mathscr{B}}\sum_{\mathfrak{p}\in\mathscr{P}_r(x)}\sum_{\substack{\tilde{E}/\mathbb{F}_p\\ \#\tilde{E}(\mathbb{F}_p)=p+1-r}} \#\mathscr{B}(\tilde{E},\mathfrak{p}) = \frac{n_K}{2}\sum_{p\in\mathcal{P}_r(x)}\frac{H(r^2-4p)}{p} + \sum_{p\in\mathcal{P}_r(x)}\mathcal{E}_p(x;\mathscr{B}), \tag{23}$$

where

$$\begin{aligned}\mathcal{E}_p(x;\mathscr{B}) := {}& \frac{H(r^2-4p)}{p^2} + \frac{H(r^2-4p)}{p\mathscr{V}_{\min}(\mathscr{B})} + p^{n_K-3/2}(\log p)^{n_K}H(r^2-4p)\\ &\times\left(\frac{1}{\mathscr{V}_1(\mathscr{B})} + \frac{1}{\mathscr{V}_2(\mathscr{B})}\right) + \frac{p^{2n_K-3/2}(\log p)^{2n_K}H(r^2-4p)}{\mathscr{V}(\mathscr{B})}.\end{aligned}$$

In [**5**, p. 178], we find the bound

$$\sum_{p\leqslant x} H(r^2-4p) \ll x^{3/2}.$$

Using this bound, together with partial summation and standard estimates, we obtain

$$\sum_{p\in\mathcal{P}_r(x)}\mathcal{E}_p(x;\mathscr{B}) \ll 1 + \frac{\sqrt{x}/\log x}{\mathscr{V}_{\min}(\mathscr{B})} + \left(\frac{1}{\mathscr{V}_1(\mathscr{B})} + \frac{1}{\mathscr{V}_2(\mathscr{B})}\right)(x\log x)^{n_K} + \frac{(x\log x)^{2n_K}}{\mathscr{V}(\mathscr{B})}. \tag{24}$$

This completes the proof of the proposition.

PROPOSITION 3. *For every $\eta > 0$,*

$$\frac{n_K}{2} \sum_{p \in \mathcal{P}_r(x)} \frac{H(r^2 - 4p)}{p} = \mathfrak{C}_{K,r,1} \pi_{1/2}(x) + O\left(\frac{\sqrt{x}}{(\log x)^{1+\eta}}\right).$$

Proof. From the definition of the Kronecker-Hurwitz class number (21) and Dirichlet's class number formula [**11**, p. 513], we have

$$\begin{aligned}\frac{n_K}{2} \sum_{p \in \mathcal{P}_r(x)} \frac{H(r^2 - 4p)}{p} &= \frac{n_K}{2} \sum_{p \in \mathcal{P}_r(x)} \sum_{\substack{k^2 \mid (r^2 - 4p) \\ d_k(p) \equiv 0, 1 \pmod 4}} \frac{\sqrt{4p - r^2}}{\pi k p} L(1, \chi_{d_k(p)}) \\ &= \frac{n_K}{\pi} \sum_{k \leqslant 2\sqrt{x}} \frac{1}{k} \sum_{p \in \mathcal{S}_k(x;r)} \frac{L(1, \chi_{d_k(p)})}{\sqrt{p}} + O\left(\sum_{p \leqslant x} \frac{\log p}{p^{3/2}} \sum_{k^2 \mid (r^2 - 4p)} 1\right),\end{aligned} \tag{25}$$

where we have used the fact that $\sqrt{4p - r^2} = 2\sqrt{p} + O(p^{-1/2})$ together with the bound $L(1, \chi_d) \ll \log d$. See [**11**, p. 120] for example. Since $\sum_{k^2 \mid (4p - r^2)} 1 \ll p^\delta$ for any $\delta > 0$, the big-O term is bounded.

Partial summation applied to the inner sum of the main term yields

$$\begin{aligned}\sum_{p \in \mathcal{S}_k(x;r)} \frac{L(1, \chi_{d_k(p)})}{\sqrt{p}} = {} & \frac{1}{\sqrt{x} \log x} \sum_{p \in \mathcal{S}_k(x;r)} L(1, \chi_{d_k(p)}) \log p \\ & + \int_{B(r)}^{x} \frac{\sum_{p \in \mathcal{S}_k(t;r)} L(1, \chi_{d_k(p)}) \log p}{2t^{3/2} \log t + t^{3/2} (\log t)^2} \, \mathrm{d}t.\end{aligned} \tag{26}$$

Substituting this back into (25), applying Proposition 1, and using (11), we have

$$\begin{aligned}\frac{n_K}{2} \sum_{p \in \mathcal{P}_r(x)} \frac{H(r^2 - 4p)}{p} &= \frac{1}{\pi \sqrt{x} \log x} A_1(x; r) - \int_{B(r)}^{x} \frac{A_1(t; r)}{2t^{3/2} \log t + t^{3/2} (\log t)^2} \, \mathrm{d}t \\ &= \frac{\mathfrak{C}_{K,r,1}}{2} \left[\frac{\sqrt{x}}{\log x} + \int_2^x \frac{\mathrm{d}t}{2\sqrt{t} \log t} + \int_2^x \frac{\mathrm{d}t}{\sqrt{t} (\log t)^2}\right] + O\left(\frac{\sqrt{x}}{(\log x)^{1+\eta}}\right) \\ &= \mathfrak{C}_{K,r,1} \pi_{1/2}(x) + O\left(\frac{\sqrt{x}}{(\log x)^{1+\eta}}\right).\end{aligned}$$

Theorem 1 now follows by combining the results of Propositions 2 and 3.

6. *The variance*

In this section, we bound the variance of $\pi_E^{r,1}(x)$. That is, we give the proof of Theorem 2 using the results of Section 4.

Proof of Theorem 2. Expanding the square and applying Theorem 1, we find that

$$\begin{aligned}\frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \left|\pi_E^{r,1}(x) - \mathfrak{C}_{K,r,1} \pi_{1/2}(x)\right|^2 = {} & \frac{1}{\#\mathscr{B}} \sum_{E \in \mathscr{B}} \left[\pi_E^{r,1}(x)\right]^2 - \left[\mathfrak{C}_{K,r,1} \pi_{1/2}(x)\right]^2 \\ & + O\left(\frac{\mathcal{E}(x; \mathscr{B}) \sqrt{x}}{\log x}\right).\end{aligned} \tag{27}$$

In the sum on the right-hand side of (27), we again expand the square and group terms according to pairs of prime ideals of equal norm and pairs of unequal norm. We

obtain

$$\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\left[\pi_E^{r,1}(x)\right]^2=\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\left[\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}=\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1+\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}\neq\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1\right]. \tag{28}$$

We bound the sum over the prime pairs of equal norm by observing that

$$\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}=\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1\leqslant\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathrm{N}\mathfrak{p}\leqslant x\\ a_{\mathfrak{p}}(E)=r\\ \deg\mathfrak{p}=1}}n_K=\frac{n_K}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\pi_E^{r,1}(x).$$

Thus, applying Theorem 1, we have

$$\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}=\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1\ll\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\pi_E^{r,1}(x)\ll\frac{\sqrt{x}}{\log x}+\mathcal{E}(x;\mathscr{B}). \tag{29}$$

For the primes of unequal norm, we argue as in the proof of Proposition 2 and write

$$\begin{aligned}\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}\neq\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1&=\sum_{\substack{\mathfrak{p},\mathfrak{p}'\in\mathscr{P}_r(x)\\ \mathrm{N}\mathfrak{p}\neq\mathrm{N}\mathfrak{p}'}}\sum_{\substack{\tilde{E}/\mathbb{F}_p,\tilde{E}'/\mathbb{F}_{p'}\\ \#E(\mathbb{F}_p)=p+1-r\\ \#E'(\mathbb{F}_{p'})=p'+1-r}}\#\mathscr{B}(E,\mathfrak{p};E',\mathfrak{p}')\\ &\quad+O\left(\frac{1}{\#\mathscr{B}}\sum_{\substack{\mathfrak{p},\mathfrak{p}'\in\mathscr{P}_r(x)\\ \mathrm{N}\mathfrak{p}\neq\mathrm{N}\mathfrak{p}'}}\sum_{\substack{E\in\mathscr{B}\\ E_{\mathfrak{p}}\text{ or }E_{\mathfrak{p}'}\text{ sing.}}}\right).\end{aligned} \tag{30}$$

As in the proof of Proposition 2, the contribution of the big-O term is negligible. Applying Lemma 2 and Theorem 5 to estimate the main term of (30), we see that the contribution from the pairs of primes of unequal norm is equal to

$$\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\sum_{\substack{\mathrm{N}\mathfrak{p},\mathrm{N}\mathfrak{p}'\leqslant x\\ \mathrm{N}\mathfrak{p}\neq\mathrm{N}\mathfrak{p}'\\ a_{\mathfrak{p}}(E)=a_{\mathfrak{p}'}(E)=r\\ \deg\mathfrak{p}=\deg\mathfrak{p}'=1}}1=\frac{n_K^2}{4}\sum_{\substack{p,p'\in\mathcal{P}_r(x)\\ p\neq p'}}\frac{H(r^2-4p)H(r^2-4p')}{pp'}+O(\mathcal{E}'(x;\mathscr{B})), \tag{31}$$

where

$$\mathcal{E}'(x;\mathscr{B}):=\frac{\sqrt{x}\log\log x}{\log x}+\frac{x/(\log x)^2}{\mathscr{V}_{\min}(\mathscr{B})}+\left(\frac{1}{\mathscr{V}_1(\mathscr{B})}+\frac{1}{\mathscr{V}_2(\mathscr{B})}\right)(x\log x)^{2n_K}+\frac{(x\log x)^{4n_K}}{\mathscr{V}(\mathscr{B})}.$$

By Proposition 3, the double sum over primes in (31) is equal to

$$\left[\frac{n_K}{2}\sum_{p\in\mathcal{P}_r(x)}\frac{H(r^2-4p)}{p}\right]^2-\frac{n_K^2}{4}\sum_{p\in\mathcal{P}_r(x)}\frac{H(r^2-4p)^2}{p^2}=\left[\mathfrak{C}_{K,r,1}\pi_{1/2}(x)\right]^2+O\left(\frac{x}{(\log x)^{2+\eta}}\right). \tag{32}$$

Combining (28), (29), (31) and (32), we have

$$\frac{1}{\#\mathscr{B}}\sum_{E\in\mathscr{B}}\left[\pi_E^{r,1}(x)\right]^2=\left[\mathfrak{C}_{K,r,1}\pi_{1/2}(x)\right]^2+O\left(\frac{x}{(\log x)^{2+\eta}}+\mathcal{E}'(x;\mathscr{B})\right).$$

Substituting this into (27), we obtain the desired result.

7. *Counting elliptic curves whose reductions are isomorphic over* $\mathbb{F}_p$

In this section, we will prove the estimates of Lemma 1 by extending standard character sum techniques as in [**8**] or [**14**]. The main difference is that we have to extend the domain of our characters to $\mathcal{O}_K$. Since we are representing $\mathcal{O}_K$ as an n_K-dimensional $\mathbb{Z}$-module, it is necessary to adapt their proof to higher dimensions.

Proof of Lemma 1 We begin by recalling that $E_{a,b}$ and $E_{a',b'}$ are isomorphic over $\mathbb{F}_p$ if and only if there exists a $u\in\mathbb{F}_p^*$ so that $a=u^4a'$ and $b=u^6b'$. Now, note that $\#\mathrm{Aut}_p(E_{a,b})=\#\{u\in\mathbb{F}_p^*: a=au^4 \text{ and } b=bu^6\}$. Therefore, we may write

$$\#\mathscr{B}(E_{a,b},\mathfrak{p})=\frac{1}{\#\mathrm{Aut}_p(E_{a,b})}\sum_{u\in(\mathcal{O}_K/\mathfrak{p})^*}\sum_{\substack{\alpha\in\mathcal{B}(\mathbf{a}_1,\mathbf{b}_1)\\ \mathfrak{p}\mid(\alpha-au^4)}}\sum_{\substack{\beta\in\mathcal{B}(\mathbf{a}_2,\mathbf{b}_2)\\ \mathfrak{p}\mid(\beta-bu^6)}}1. \tag{33}$$

Refer back to Equation (5) for the definition of $\mathcal{B}(\mathbf{a},\mathbf{b})$. To estimate this sum, we write

$$\#\mathscr{B}(E_{a,b},\mathfrak{p})=\frac{1}{\#\mathrm{Aut}_p(E_{a,b})}\sum_{u\in(\mathcal{O}_K/\mathfrak{p})^*}\sum_{\substack{\alpha\in\mathcal{B}(\mathbf{a}_1,\mathbf{b}_1)\\ \beta\in\mathcal{B}(\mathbf{a}_2,\mathbf{b}_2)}}\frac{1}{p^2}\sum_{(\psi,\psi')}\psi(\alpha-au^4)\psi'(\beta-bu^6), \tag{34}$$

where the innermost sum is over all pairs (ψ,ψ') of additive characters on $\mathcal{O}_K/\mathfrak{p}\cong\mathbb{F}_p$. The main term results from when $\psi=\psi'=\psi_0$, the trivial character, which contributes $((p-1)/p^2\#\mathrm{Aut}_p(E_{a,b}))\mathscr{V}(\mathscr{B})+O((\mathscr{V}(\mathscr{B}))/p\mathscr{V}_{\min}(\mathscr{B})))$. The remaining terms are bounded by

$$\frac{1}{p^2\#\mathrm{Aut}_p(E_{a,b})}\sum_{(\psi,\psi')\neq(\psi_0,\psi_0)}\left|\sum_{u\in(\mathcal{O}_K/\mathfrak{p})^*}\overline{\psi}(au^4)\overline{\psi'}(bu^6)\right|\left|\sum_{\alpha\in\mathcal{B}(a_1,b_1)}\psi(\alpha)\right|\left|\sum_{\beta\in\mathcal{B}(a_2,b_2)}\psi(\beta)\right|. \tag{35}$$

Since in the line above at least one of ψ and ψ' is not trivial, we will assume for the moment that it is ψ. Using well-known facts about additive characters modulo p, we may write

$$\overline{\psi}(au^4)\overline{\psi'}(bu^6)=\overline{\psi}(au^4+mbu^6)$$

for some $m\in\mathbb{F}_p$. We think of the expression au^4+mbu^6 as a polynomial in u over $\mathcal{O}_K/\mathfrak{p}\cong\mathbb{F}_p$ of degree either 4 or 6. Thus, since $\mathfrak{p}\nmid 6$, we may apply Weil's Theorem [**18**, p. 223], which yields

$$\left|\sum_{u\in(\mathcal{O}_K/\mathfrak{p})^*}\overline{\psi}(au^4+mbu^6)\right|\ll\sqrt{p}.$$

We now estimate $\sum_\psi|\sum_{\alpha\in\mathcal{B}(a_1,b_1)}\psi(\alpha)|$. If $\psi=\psi_0$, then $|\sum_{\alpha\in\mathcal{B}(a_1,b_1)}\psi(\alpha)|\ll\mathscr{V}_1(\mathscr{B})$ and it remains to bound $\sum_{\psi\neq\psi_0}|\sum_{\alpha\in\mathcal{B}(a_1,b_1)}\psi(\alpha)|$. We now write each α in terms of our

fixed basis $\mathcal{B} = \{\gamma_1, \ldots, \gamma_{n_K}\}$ as $\alpha = \sum_{j=1}^{n_K} c_j \gamma_j$. Since we assumed that $\mathfrak{p} \nmid \prod_{j=1}^{n_K} \gamma_j$, each γ_j is nonzero modulo $\mathfrak{p}$. Thus,

$$\begin{aligned}
\sum_{\psi \neq \psi_0} \left| \sum_{\alpha \in \mathcal{B}(a_1,b_1)} \psi(\alpha) \right| &= \sum_{\psi \neq \psi_0} \left| \sum_{\alpha \in \mathcal{B}(a_1,b_1)} \prod_{j=1}^{n_K} \psi(c_j \gamma_j) \right| \\
&= \sum_{\psi \neq \psi_0} \prod_{j=1}^{n_K} \left| \sum_{c_j = \lceil a_{1,j} - b_{1,j} \rceil}^{\lfloor a_{1,j} + b_{1,j} \rfloor} \psi(c_j \gamma_j) \right| \\
&\leqslant \prod_{j=1}^{n_K} \sum_{\psi \neq \psi_0} \left| \sum_{c_j = \lceil a_{1,j} - b_{1,j} \rceil}^{\lfloor a_{1,j} + b_{1,j} \rfloor} \psi(c_j \gamma_j) \right| \\
&= \prod_{j=1}^{n_K} \sum_{\psi \neq \psi_0} \left| \sum_{c_j = \lceil a_{1,j} - b_{1,j} \rceil}^{\lfloor a_{1,j} + b_{1,j} \rfloor} \psi(c_j) \right| \\
&\ll p^{n_K} (\log p)^{n_K}.
\end{aligned}$$

The same line of reasoning suffices to estimate $\sum_{\psi} |\sum_{\beta \in \mathcal{B}(a_2,b_2)} \psi(\beta)|$. The result now follows by appropriately combining all of these estimates.

8. *Averaging special values of Dirichlet L-functions*

In this section, we sketch the proof of Proposition 1. That is, we show how to compute

$$A_1(x; r) = n_K \sum_{k \leqslant 2\sqrt{x}} \frac{1}{k} \sum_{p \in \mathcal{S}_k(x;r)} L(1, \chi_{d_k(p)}) \log p. \tag{36}$$

Similar averages of the special values of Dirichlet L-functions arise in previous work on the average Lang–Trotter problem. The general strategy has been to reorder the summation so that one arrives at a sum that can be easily estimated using the Prime Number Theorem for primes in arithmetic progressions. For the case of degree 1 primes of a number field K, one needs to estimate sums of the form

$$\sum_{\substack{p \leqslant x \\ p \text{ splits comp. in } K \\ p \equiv a \pmod q}} \log p$$

for essentially every possible value of a and q.

When $K/\mathbb{Q}$ is Abelian, as in [**3**], the condition that p splits completely in K is determined by congruence conditions. That is, there exists an integer m_K and a subgroup $G_{m_K} \subseteq (\mathbb{Z}/m_K\mathbb{Z})^*$ so that p splits completely in K if and only if $p \equiv b \pmod{m_K}$ for some $b \in G_{m_K}$. One may check that this definition of G_{m_K} agrees with the one given in Section 3 in the case that $K = \mathcal{A}$ is an Abelian extension of $\mathbb{Q}$. Thus, if $K/\mathbb{Q}$ is Abelian, one may rewrite the above sum as

$$\sum_{\substack{p \leqslant x \\ p \text{ splits comp. in } K \\ p \equiv a \pmod q}} \log p = \sum_{b \in G_{m_K}} \sum_{\substack{p \leqslant x \\ p \equiv b \pmod{m_K} \\ p \equiv a \pmod q}} \log p.$$

The inner sum can then be estimated by the Prime Number Theorem for primes in arithmetic progressions when $a \equiv b \pmod{(q, m_K)}$. Otherwise, the sum is empty.

When $K/\mathbb{Q}$ is a non-Abelian Galois extension, one cannot write down a list of congruence conditions that determine exactly when a rational prime will split completely in K. Essentially, the remedy is apply the Chebotarëv Density Theorem to the appropriate Galois extension of K. In order to transform our sum into a form appropriate for application of the Chebotarëv Density Theorem, we make the following simple observation. For each rational prime p splitting completely in K, there are exactly n_K primes $\mathfrak{p}$ of K lying above p, all satisfying $\mathrm{N}\mathfrak{p} = p$. Therefore,

$$\sum_{\substack{p \leqslant x \\ p \text{ splits comp. in } K \\ p \equiv a \pmod q}} \log p = \frac{1}{n_K} \sum_{\substack{\mathrm{N}\mathfrak{p} \leqslant x \\ \deg \mathfrak{p} = 1 \\ \mathrm{N}\mathfrak{p} \equiv a \pmod q}} \log \mathrm{N}\mathfrak{p} + O(1). \tag{37}$$

The sum on the right-hand side is now in appropriate form to be estimated by the Chebotarëv Density Theorem. We now explain this application.

For each positive integer q, let ζ_q be a primitive qth root of unity and let G_q denote the image of the natural map

$$\mathrm{G}al(K(\zeta_q)/K) \hookrightarrow \mathrm{G}al(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/q\mathbb{Z})^*. \tag{38}$$

Thus, for each positive integer q, we have a canonical identification of $\mathrm{G}al(K(\zeta_q)/K)$ with a certain subgroup of $(\mathbb{Z}/q\mathbb{Z})^*$, which we denote by G_q. For the case that $q = m_K$, it is easy to check that this definition of G_{m_K} agrees with the one given in Section 3. Indeed, $G_{m_K} \cong \mathrm{G}al(K(\zeta_{m_K})/K) \cong \mathrm{G}al(\mathbb{Q}(\zeta_{m_K})/\mathcal{A})$.

For each prime ideal $\mathfrak{p}$ of K not ramifying in $K(\zeta_q)$, it is easy to check that the Frobenius automorphism at $\mathfrak{p}$ is determined by the residue of $\mathrm{N}\mathfrak{p}$ modulo q. Thus, provided that a is in the image of the natural map (38), it follows from Chebotarëv Density Theorem that

$$\sum_{\substack{\mathrm{N}\mathfrak{p} \leqslant x \\ \mathrm{N}\mathfrak{p} \equiv a \pmod q}} \log \mathrm{N}\mathfrak{p} \sim \frac{1}{\varphi_K(q)} x,$$

where $\varphi_K(q) := \#G_q$. Otherwise, the sum is empty. Since the contribution from the primes $\mathfrak{p}$ of degree greater than or equal to 2 is $O(\sqrt{x})$ (with an implied constant depending on K, but not on q), it also follows that

$$\sum_{\substack{\mathrm{N}\mathfrak{p} \leqslant x \\ \deg \mathfrak{p} = 1 \\ \mathrm{N}\mathfrak{p} \equiv a \pmod q}} \log \mathrm{N}\mathfrak{p} \sim \frac{1}{\varphi_K(q)} x.$$

Applying this in (37) yields the asymptotic identity

$$\sum_{\substack{p \leqslant x \\ p \text{ splits comp. in } K \\ p \equiv a \pmod q}} \log p = \frac{1}{n_K} \sum_{\substack{\mathrm{N}\mathfrak{p} \leqslant x \\ \deg \mathfrak{p} = 1 \\ \mathrm{N}\mathfrak{p} \equiv a \pmod q}} \log \mathrm{N}\mathfrak{p} \sim \frac{1}{n_K \varphi_K(q)} x.$$

Essentially, by replacing our sum over rational primes by the appropriate sum over prime ideals of K and noting that the degree 1 primes comprise a density 1 subset of the set of all prime ideals of K, we are free to "ignore" the condition on the degree. Note that this trick will not work if one wants to count degree 2 primes satisfying congruence conditions.

We will need the following result in order to control the error incurred by invoking the Chebotarëv Density Theorem to estimate sums of the form

$$\theta_K(x;1,q,a) := \sum_{\substack{\mathrm{N}\mathfrak{p}\leqslant x\\ \deg\mathfrak{p}=1\\ \mathrm{N}\mathfrak{p}\equiv a \pmod q}} \log \mathrm{N}\mathfrak{p}.$$

THEOREM 6. *For any $M>0$,*

$$\sum_{q\leqslant Q}\sum_{a\in G_q}\left(\theta_K(x;1,q,a)-\frac{x}{\varphi_K(q)}\right)^2 \ll xQ\log x,$$

provided that $x(\log x)^{-M}\leqslant Q\leqslant x$.

Remark. This result is a slight modification of the main result of [**21**] and can be proved similarly with only minor alterations to the proof. In [**21**], the main result is stated with $\theta_K(x;1,q,a)$ replaced by the Chebychev function

$$\psi_K(x;q,a) := \sum_{\substack{\mathrm{N}\mathfrak{p}^m\leqslant x\\ \mathrm{N}\mathfrak{p}^m\equiv a \pmod q}} \log \mathrm{N}\mathfrak{p}.$$

The key observation when altering the proof is that the contribution from prime powers and higher degree primes is negligible.
Remark. It should also be possible to achieve an asymptotic version of this result along the same lines as [**22**].

Proof of Proposition 1. As in [**6**, p. 193], we introduce a parameter U (to be chosen later) and begin with the identity

$$L(1,\chi_{d_k(p)}) = \sum_{n\geqslant 1}\left(\frac{d_k(p)}{n}\right)\frac{1}{n} = \sum_{n\geqslant 1}\left(\frac{d_k(p)}{n}\right)\frac{e^{-n/U}}{n} + O\left(\frac{|d_k(p)|^{7/32}}{U^{1/2}}\right).$$

Whence, if

$$U \geqslant x^{7/16}(\log x)^{2\eta}, \tag{39}$$

then substitution and interchanging sums yields

$$A_1(x;r) = n_K \sum_{k\leqslant 2\sqrt{x}}\frac{1}{k}\sum_{n\geqslant 1}\frac{e^{-n/U}}{n}\sum_{p\in\mathcal{S}_k(x;r)}\left(\frac{d_k(p)}{n}\right)\log p + O\left(\frac{x}{(\log x)^{\eta}}\right). \tag{40}$$

We now introduce another parameter V and observe that contribution to (40) from the "large" values of k is

$$\sum_{V<k\leqslant 2\sqrt{x}}\frac{1}{k}\sum_{n\geqslant 1}\frac{e^{-n/U}}{n}\sum_{p\in\mathcal{S}_k(x;r)}\left(\frac{d_k(p)}{n}\right)\log p \ll (x\log x)V^{-2}\log U \ll \frac{x}{(\log x)^{\eta}}$$

if

$$V \geqslant (\log x)^{(\eta+2)/2}, \tag{41}$$

$$U \leqslant x. \tag{42}$$

We also observe that for U satisfying (39), the contribution from the "large" n is

$$\sum_{k\leqslant V}\frac{1}{k}\sum_{n\geqslant U\log U}\frac{e^{-n/U}}{n}\sum_{p\in\mathcal{S}_k(x;r)}\left(\frac{d_k(p)}{n}\right)\log p \ll \frac{\log x}{U\log U}\sum_{k\leqslant V}\frac{1}{k}\sum_{\substack{m\leqslant x\\ k^2|(4m-r^2)}}1 \ll \frac{x}{(\log x)^{\eta}}. \tag{43}$$

Therefore, we have

$$A_1(x;r) = n_K\sum_{k\leqslant V}\frac{1}{k}\sum_{n\leqslant U\log U}\frac{e^{-n/U}}{n}\sum_{p\in\mathcal{S}_k(x;r)}\left(\frac{d_k(p)}{n}\right)\log p + O\left(\frac{x}{(\log x)^{\eta}}\right). \tag{44}$$

Now recall the notation of Section 4·2. In particular, recall that the definition of $\mathscr{P}_r(x)$ explicitly excludes any prime ideals $\mathfrak{p}$ which ramify in the fixed Galois extension $K(\zeta_{m_K})/K$. If $\mathfrak{p}$ is a prime of K that does not ramify in $K(\zeta_{m_K})$, then it follows (by calculating the Frobenius at $\mathfrak{p}$ and applying the map (38)) that $\mathrm{N}\mathfrak{p}\equiv b \pmod{m_K}$ for some $b\in G_{m_K}$. For each pair n, k, we regroup the terms of the innermost sum in (44) to see that

$$\begin{aligned} n_K\sum_{p\in\mathcal{S}_k(x;r)}\left(\frac{d_k(p)}{n}\right)\log p &= \sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z})\\ a\equiv 0,1 \pmod 4}}\left(\frac{a}{n}\right)\sum_{\substack{p\in\mathcal{S}_k(x;r)\\ d_k(p)\equiv a \pmod{4n}}} n_K\log p\\ &= \sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z})\\ a\equiv 0,1 \pmod 4}}\left(\frac{a}{n}\right)\sum_{b\in G_{m_K}}\sum_{\substack{p\in\mathcal{S}_k(x;r)\\ d_k(p)\equiv a \pmod{4n}\\ p\equiv b \pmod{m_K}}} n_K\log p\\ &= \sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z})\\ a\equiv 0,1 \pmod 4\\ 4|(r^2-ak^2)}}\left(\frac{a}{n}\right)\sum_{b\in G_{m_K}}\sum_{\substack{\mathfrak{p}\in\mathscr{P}_r(x)\\ \mathrm{N}\mathfrak{p}\equiv\frac{r^2-ak^2}{4} \pmod{nk^2}\\ \mathrm{N}\mathfrak{p}\equiv b \pmod{m_K}}}\log\mathrm{N}\mathfrak{p}. \end{aligned} \tag{45}$$

Note that if $4|(r^2-ak^2)$ and $(r^2-ak^2)/4$ is not coprime to nk^2, then there can be at most finitely many degree one prime ideals $\mathfrak{p}$ satisfying the two conditions on the innermost sum in the last line of (45). Furthermore, this can only happen when the greatest common divisor of nk^2 and the least positive residue of $(r^2-ak^2)/4$ is itself a prime, say ℓ, and $\mathrm{N}\mathfrak{p}=\ell$. Now, if ℓ is an odd prime dividing k, then ℓ^2 divides both $(r^2-ak^2)/4$ and nk^2. Thus, this situation can only arise from 2 and those primes dividing n. Whence the last line of (45) is equal to

$$\sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z})\\ a\equiv 0,1 \pmod 4\\ (r^2-ak^2,4nk^2)=4}}\left(\frac{a}{n}\right)\sum_{b\in G_{m_K}}\sum_{\substack{\mathfrak{p}\in\mathscr{P}_r(x)\\ \mathrm{N}\mathfrak{p}\equiv\frac{r^2-ak^2}{4} \pmod{nk^2}\\ \mathrm{N}\mathfrak{p}\equiv b \pmod{m_K}}}\log\mathrm{N}\mathfrak{p} + O\left(\sum_{\substack{\ell|n\\ \ell \text{ prime}}}\log\ell\right). \tag{46}$$

Now, we interchange the outer two sums and note that the two conditions on the innermost sum are contradictory unless $4b\equiv r^2-ak^2 \pmod{(4m_K, 4nk^2)}$. Therefore, the main term of (46) is equal to

$$\sum_{b\in G_{m_K}}\sum_{\substack{a\in(\mathbb{Z}/4n\mathbb{Z})\\ a\equiv 0,1 \pmod 4\\ (r^2-ak^2,4nk^2)=4\\ 4b\equiv r^2-ak^2 \pmod{(4m_K,4nk^2)}}}\left(\frac{a}{n}\right)\sum_{\substack{\mathrm{N}\mathfrak{p}\leqslant x\\ \deg\mathfrak{p}=1\\ \mathrm{N}\mathfrak{p}\equiv\frac{r^2-ak^2}{4} \pmod{nk^2}\\ \mathrm{N}\mathfrak{p}\equiv b \pmod{m_K}}}\log\mathrm{N}\mathfrak{p} + O(1/k^2), \tag{47}$$

where the big-O accounts for the prime ideals with norm less than $B(r)$ and those dividing the different of $K/\mathbb{Q}$. See the definitions in Section 4·2. Now, the two conditions on the

innermost sum are equivalent via the Chinese Remainder Theorem to a single condition modulo the least common multiple $[m_K, nk^2]$. Therefore, (44), (45), (46), (47) and the Chebotarëv Density Theorem applied the prime ideals of K satisfying the conditions of the innermost sum of (47) imply that

$$\begin{aligned} A_1(x;r) = x \sum_{b\in G_{m_K}} \sum_{\substack{n\leqslant U\log U,\\ k\leqslant V}} \frac{e^{-n/U} c_k^{r,b,m_K}(n)}{nk\varphi_K([m_K, nk^2])} + O\left(\frac{x}{(\log x)^\eta}\right) \\ + O\left(\sum_{k\leqslant V}\frac{1}{k}\sum_{n\leqslant U\log U}\sum_{h\in G_{[m_K,nk^2]}} \frac{e^{-n/U}}{n}|E_K(x;1,[m_K,nk^2],h)|\right), \end{aligned} \tag{48}$$

where $c_k^{r,b,m_K}(n)$ is the function defined by equation (13), and for $h \in G_q$

$$E_K(x;1,q,h) := \theta_K(x;1,q,h) - \frac{x}{\varphi_K(q)}.$$

Facts from Galois theory imply that $G_{[m_K,nk^2]}$ is a quotient of $G_{m_K nk^2}$; whence, via the triangle inequality, we have

$$\sum_{h\in G_{[m_K,nk^2]}} |E_K(x;1,[m_K,nk^2],h)| \leqslant \sum_{h\in G_{m_K nk^2}} |E_K(x;1,m_K nk^2,h)|.$$

We now choose

$$\begin{aligned} U &= \frac{x}{(\log x)^{5\eta+15}}, \\ V &= (\log x)^{(\eta+3)/2} \end{aligned}$$

and note that this choice is in accordance with (39), (41) and (42). Thus, by the Cauchy–Schwarz inequality and Theorem 6, we have

$$\begin{aligned} &\sum_{k\leqslant V}\frac{1}{k}\left[\sum_{n\leqslant U\log U}\sum_{h\in G_{[m_K,nk^2]}}\frac{e^{-n/U}}{n}|E_K(x;1,[m_K,nk^2],h)|\right] \\ &\quad\leqslant \sum_{k\leqslant V}\frac{1}{k}\left[\sum_{n\leqslant U\log U}\frac{\varphi_K(m_K nk^2)}{n^2}\right]^{1/2}\left[\sum_{n\leqslant U\log U}\sum_{h\in G_{m_K nk^2}} E_K(x;1,m_K nk^2,h)^2\right]^{1/2} \\ &\quad\ll V\sqrt{\log U}\left[\sum_{q\leqslant m_K V^2 U\log U}\sum_{h\in G_q} E_K(x;1,q,h)^2\right]^{1/2} \\ &\quad\ll V\sqrt{\log U}\sqrt{xV^2U\log U\log x} \\ &\quad\ll \tfrac{x}{(\log x)^\eta} \end{aligned}$$

since $x(\log x)^{-M} \leqslant V^2 U\log U \leqslant x$, say with $M = 4\eta + 11$. Therefore, equation (48) becomes

$$A_1(x;r) = x\sum_{b\in G_{m_K}}\sum_{\substack{n\leqslant U\log U,\\ k\leqslant V}}\frac{e^{-n/U}c_k^{r,b,m_K}(n)}{nk\varphi_K([m_K,nk^2])} + O\left(\frac{x}{(\log x)^\eta}\right). \tag{49}$$

Recall the definition of $\mathcal{A}$ from the first paragraph of Section 3, and observe that since $m_K|[m_K, nk^2]$, we have

$$\mathcal{A} = \mathbb{Q}^{\mathrm{cyc}} \cap K = \mathbb{Q}(\zeta_{m_K}) \cap K \subseteq \mathbb{Q}(\zeta_{[m_K,nk^2]}).$$

Thus, we have the isomorphism $G_{[m_K,nk^2]} \cong Gal(\mathbb{Q}(\zeta_{[m_K,nk^2]})/\mathcal{A})$, and recalling the definition of $n_{\mathcal{A}}$, we have the identity

$$\varphi([m_K, nk^2]) = n_{\mathcal{A}}\varphi_K([m_K, nk^2]).$$

Hence, equation (49) becomes

$$A_1(x; r) = xn_{\mathcal{A}} \sum_{b\in G_{m_K}} \sum_{\substack{n\leqslant U\log U,\\ k\leqslant V}} \frac{e^{-n/U}c_k^{r,b,m_K}(n)}{nk\varphi([m_K, nk^2])} + O\left(\frac{x}{(\log x)^{\eta}}\right). \tag{50}$$

The final step of the proof is to show that

$$\sum_{b\in G_{m_K}} \sum_{\substack{n\leqslant U\log U,\\ k\leqslant V}} \frac{e^{-n/U}c_k^{r,b,m_K}(n)}{nk\varphi([m_K, nk^2])} = \sum_{b\in G_{m_K}} \sum_{\substack{n\geqslant 1,\\ k\geqslant 1}} \frac{c_k^{r,b,m_K}(n)}{nk\varphi([m_K, nk^2])} + O\left(\frac{1}{(\log x)^{\eta}}\right)$$

which, as a side effect, demonstrates the absolute convergence of the double infinite sum in (12). This is done in a manner similar to [**6**, pp. 197–199].

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REFERENCES

[**1**] S. BAIER. The Lang-Trotter conjecture on average. *J. Ramanujan Math. Soc.* **22**(4): (2007), 299–314.

[**2**] J. BATTISTA, J. BAYLESS, D. IVANOV and K. JAMES. Average Frobenius distributions for elliptic curves with nontrivial rational torsion. *Acta Arith.* **119**(1) (2005), 81–91.

[**3**] N. CALKIN, B. FAULKNER, K. JAMES, M. KING and D. PENNISTON. Average Frobenius distributions for elliptic curves over Abelian extensions. *Acta Arith.* (to appear).

[**4**] A. C. COJOCARU and M. RAM MURTY. *An Introduction to Sieve Methods and Their Applications*, London Mathematical Society Student Texts. vol. 66 (Cambridge University Press, 2006).

[**5**] C. DAVID and F. PAPPALARDI. Average Frobenius distributions of elliptic curves. *Internat. Math. Res. Notices* **1999**(4) (1999), 165–183.

[**6**] C. DAVID and F. PAPPALARDI. Average Frobenius distribution for inerts in $\mathbb{Q}(i)$. *J. Ramanujan Math. Soc.* **19**(3) (2004), 181–201.

[**7**] M. DEURING. Die Typen der Multiplikatorenringe elliptischer Funktionenkörper. *Abh. Math. Sem. Hansischen Univ.* **14** (1941), 197–272.

[**8**] E. FOUVRY and M. RAM MURTY. On the distribution of supersingular primes. *Canad. J. Math.* **48**(1) (1996), 81–104.

[**9**] E. HECKE. Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. *Math. Z.* **1**(4) (1918), 357–376.

[**10**] E. HECKE. Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. *Math. Z.* **6**(1-2) (1920), 11–51.

[**11**] H. IWANIEC and E. KOWALSKI. *Analytic Number Theory*. American Mathematical Society Colloquium Publications. vol. 53 (American Mathematical Society, 2004).

[**12**] K. JAMES. Average Frobenius distributions for elliptic curves with 3-torsion. *J. Number Theory* **109**(2) (2004), 278–298.

[**13**] K. JAMES. Averaging special values of Dirichlet L-series. *Ramanujan J.* **10**(1) (2005), 75–87.

[**14**] K. JAMES and G. YU. Average Frobenius distribution of elliptic curves. *Acta Arith.* **124**(1) (2006), 79–100.

[**15**] S. LANG. *Algebraic Number Theory.* Graduate Texts in Mathematics. vol. 110 (Springer-Verlag, 1994).

[**16**] S. LANG and H. TROTTER. *Frobenius Distributions in* GL_2*-extensions*. Lecture Notes in Mathematics, Vol. 504. (Springer-Verlag, 1976). Distribution of Frobenius automorphisms in GL_2-extensions of the rational numbers.

[**17**] H.W. LENSTRA, JR. Factoring integers with elliptic curves. *Ann. of Math.* (2), **126**(3) (1987), 649–673.

[**18**] R. LIDL and H. NIEDERREITER. *Finite Fields*. Encyclopedia of Mathematics and its Applications. vol. 20 (Cambridge University Press, 1997). With a foreword by P. M. Cohn.

[**19**] R. SCHOOF. Nonsingular plane cubic curves over finite fields. *J. Combin. Theory Ser. A* **46**(2) (1987), 183–211.

[**20**] J. H. SILVERMAN. *The Arithmetic of Elliptic Curves* (Springer-Verlag, 1986).

[**21**] E. SMITH. A generalization of the Barban-Davenport-Halberstam Theorem to number fields. *J. Number Theory* **129**(11) (2009), 2735–2742.

[**22**] E. SMITH. A Barban-Davenport-Halberstam asymptotic for number fields. *Proc. Amer. Math. Soc.* **138**(7) (2010), 2301–2309.

[**23**] L. C. WASHINGTON. *Introduction to Cyclotomic Fields.* Graduate Texts in Mathematics. vol. 83 (Springer-Verlag, 1997).