

AVERAGE FROBENIUS DISTRIBUTION OF ELLIPTIC CURVES

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ABSTRACT. The Sato-Tate conjecture asserts that given an elliptic curve without complex multiplication, the primes whose Frobenius elements have their trace in a given interval $(2\alpha\sqrt{p}, 2\beta\sqrt{p})$ have density given by $\frac{2}{\pi} \int_{\alpha}^{\beta} \sqrt{1-t^2} dt$. We prove that this conjecture is true on average in a more general setting.

1. INTRODUCTION

B . Given an elliptic curve E/\mathbb{F}_p , let $E(\mathbb{F}_p)$ denote the Mordell-Weil group, which consists of the \mathbb{F}_p -rational points on the curve along with an identity at infinity. A simple heuristic shows that the normal order of $E(\mathbb{F}_p)$ is $p+1$. If letting

$$a_E(p) := p + 1 - |E(\mathbb{F}_p)|,$$

Hasse's Theorem asserts that

$$|a_E(p)| < 2\sqrt{p}. \tag{1.1}$$

It follows from Deuring's Theorem (see [2] or [3]) that (1.1) is best possible in the sense that, given a prime p , and an integer $r \in (-2\sqrt{p}, 2\sqrt{p})$, there exists an elliptic curve E/\mathbb{F}_p such that $a_E(p) = r$.

Let

$$\frac{a_E(p)}{2\sqrt{p}} = \cos \theta_E(p), \quad \theta_E(p) \in [0, \pi]^1.$$

Then given E/\mathbb{Q} , it is quite natural to ask how $\theta_E(p)$ varies with p .

When E has complex multiplication, the answer turns out to be easy. In this case, asymptotically, half of primes p satisfy $a_E(p) = 0$. Apart from these *supersingular* primes, the primes p with $\theta_E(p)$ in a fixed range are those given by $p = f(u, v)$, where $f(u, v)$ is a certain positive definite binary quadratic form, with $|u|/p$ in the corresponding range. The distribution of such

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¹We remark that, in some literatures, it is defined $\frac{-a_E(p)}{2\sqrt{p}} = \cos \theta_E(p)$. This however does not change the statement of the Sato-Tate Conjecture.

primes, with a certain uniformity, is known to us and can be traced back to Hecke's famous work [6, 7].

When E does not have complex multiplication, the problem is much more difficult. With the experimental support of Sato, Tate [9] has given theoretical evidence for the following conjecture (for further discussion see [2, 4, 8]).

Sato-Tate Conjecture: *Suppose E is an elliptic curve over \mathbb{Q} which does not admit complex multiplication. For any $0 \leq \theta_1 \leq \theta_2 \leq \pi$, and $x > 1$, let*

$$\pi_E^{(\theta_1, \theta_2)}(x) := \#\{p \leq x : \theta_1 \leq \theta_E(p) \leq \theta_2\},$$

then

$$\lim_{x \rightarrow \infty} \frac{\pi_E^{(\theta_1, \theta_2)}(x)}{\pi(x)} = \frac{2}{\pi} \int_{\theta_1}^{\theta_2} \sin^2 \theta d\theta,$$

where $\pi(x)$ is the number of primes up to x .

By a change of variables, it is clear that, for $-1 \leq \alpha \leq \beta \leq 1$, and $x > 1$, if letting

$$\pi_E(\alpha, \beta; x) := \#\left\{p \leq x : \alpha \leq \frac{a_E(p)}{2\sqrt{p}} \leq \beta\right\},$$

then the Sato-Tate Conjecture is equivalent to

$$\lim_{x \rightarrow \infty} \frac{\pi_E(\alpha, \beta; x)}{\pi(x)} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sqrt{1-t^2} dt.$$

In [2], using the Selberg trace formula, Birch proved that, for any positive integer k , one has

$$\text{mean}|a_E(p)|^{2k} \sim \frac{2k!p^k}{k!(k+1)!}$$

as $p \rightarrow \infty$. Here the mean is taken subject to E varying over all elliptic curves over \mathbb{F}_p . Birch's result essentially implies that the Sato-Tate Conjecture for elliptic curves is true on average.

Suppose \mathcal{S} is a subset of \mathbb{Z} which is not too sparse. For $-1 \leq \alpha \leq \beta \leq 1$, and $x > 1$, let $\pi_E(\alpha, \beta, \mathcal{S}; x)$ be the number of primes $p \leq x$ satisfying

$$\alpha \leq \frac{a_E(p)}{2\sqrt{p}} \leq \beta, \quad a_E(p) \in \mathcal{S}.$$

From the probabilistic point of view, there should be an asymptotic formula for $\pi_E(\alpha, \beta, \mathcal{S}; x)$ as $x \rightarrow \infty$ if \mathcal{S} is not too sparse and is uniformly distributed. It is then natural to ask how $\pi_E(\alpha, \beta, \mathcal{S}; x)$ may depend on \mathcal{S} .

We can show that, when \mathcal{S} is nice enough, $\pi_{E(a,b)}(\alpha, \beta, \mathcal{S}; x)$ has an asymptotic formula for almost all elliptic curves $E(a, b)$ with a, b varying in certain ranges depending on x , where $E(a, b)$

is given by the equation

$$E(a, b) : \quad y^2 = x^3 + ax + b.$$

This gives a heuristic for the relation of $\pi_E(\alpha, \beta, \mathcal{S}; x)$ and \mathcal{S} .

Due to symmetry, we may consider only the average behavior of

$$\pi_E(\alpha, \mathcal{S}; x) := \pi_E(0, \alpha, \mathcal{S}; x)$$

for a given \mathcal{S} and $\alpha \in [0, 1]$. Without loss of generality, \mathcal{S} may be taken as a subset of $\mathbb{Z}_{\geq 0}$.

It will be clear from our treatment that \mathcal{S} is *nice* for our purpose provided that, for large N , the exponential sum

$$\sum_{n \in \mathcal{S} \cap [1, N]} e(n^2 \lambda)$$

can be well approximated when λ is on major arcs (which are reasonably large in terms of N) in applying the Hardy-Littlewood circle method. From this, experts who are familiar with the Hardy-Littlewood method may have seen that \mathcal{S} can be chosen as, for example, an arithmetic progression, the set of k -th powers, smooth numbers, values of an integer-valued polynomial at primes, or even the set of integers which are sums of a fixed number of exponentials (powers of 2, for instance).

In this paper, we shall only consider the case that \mathcal{S} is the set of all k -th powers. The results for various cases listed above may follow from similar proofs. Henceforth, \mathcal{K} always stands for the set of all k -th powers for a fixed $k \in \mathbb{N}$.

For $0 < \alpha < 1$, and positive real numbers U, V, A, B, X , let

$$S_\alpha(U, V, A, B; \mathcal{K}; X) = \frac{1}{AB} \sum_{\substack{U < a \leq U+A \\ V < b \leq V+B}} \pi_{E(a,b)}(\alpha, \mathcal{K}; X).$$

The function $S_\alpha(U, V, A, B; \mathcal{K}; X)$ measures, in a certain range, the average number of primes up to X with $a_E(p) \in \mathcal{K}$ and $\theta_E(p)$ belonging to a given interval $[\arccos \alpha, \pi/2)$. We will investigate the asymptotic behavior of $S_\alpha(U, V, A, B; \mathcal{K}; X)$ when A and B are large enough in comparison with X , as $X \rightarrow \infty$.

Theorem 1. *Let $0 < \alpha < 1$ be fixed and let U and V be any real numbers. For any given $\varepsilon > 0$ and X sufficiently large, if $A, B > X \log X$, then we have*

$$S_\alpha(U, V, A, B; \mathcal{K}; X) \sim c_k(\alpha) \pi_k(X),$$

where

$$\pi_k(X) = \int_2^X \frac{t^{\frac{1}{2k}-\frac{1}{2}}}{\log t} dt = (1 + O((\log X)^{-1})) \frac{2k}{k+1} \frac{X^{\frac{1}{2}+\frac{1}{2k}}}{\log X},$$

and

$$c_k(\alpha) = \left(\frac{1}{3} + \frac{2}{3}\delta(k) \right) \frac{2^{\frac{1}{k}}}{k\pi} \int_0^\alpha |t|^{\frac{1}{k}-1} \sqrt{1-t^2} dt,$$

where $\delta(k)$ is given by

$$\delta(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \quad (1.2)$$

Note that, taking $k = 1$, Theorem (1) yields that the Sato-Tate Conjecture holds on average. With a little extra effort, we can show that $c_k(\alpha)\pi_k(X)$ is actually the normal order of $\pi_E(\alpha, \mathcal{K}; X)$.

Theorem 2. *Let $0 < \alpha < 1$ be fixed, U, V be any real numbers, A, B, X be sufficiently large real numbers and $A, B > (X \log X)^2$. Then we have*

$$\frac{1}{AB} \sum_{\substack{U < a \leq U+A \\ V < b \leq V+B}} \left| \pi_{E(a,b)}(\alpha, \mathcal{K}; X) - c_k(\alpha)\pi_k(X) \right|^2 = o((\pi_k(X))^2).$$

We also note that under our assumptions on A and B , the contribution to $S_\alpha(U, V, A, B; \mathcal{K}; X)$ by curves $E(a, b)$ with complex multiplication is negligible. (There are only 13 j -invariants associated with CM curves, hence, the total number of CM curves encountered is $O(A+B)$ and therefore the contribution of CM curves to $S_\alpha(U, V, A, B; \mathcal{K}; X)$ is easily seen to be $O(\pi_k(X)(\frac{1}{A} + \frac{1}{B}))$ which under our assumptions on A and B is $o(1)$.) In view of this and Theorem 2, it seems reasonable for us to make the following conjecture.

Conjecture 1. *For a given elliptic curve E/\mathbb{Q} without complex multiplication, and $-1 < \alpha < \beta < 1$, we have*

$$\pi_E(\alpha, \beta, \mathcal{K}; X) \sim (c_k(\beta) - c_k(\alpha))\pi_k(X). \quad (1.3)$$

We remark that the special case of (1.3) ($k = 2$) has been considered in [?]

We have not pursued uniformity in α in Theorems 1 and 2. Nevertheless, it is clear from our proof that α can be related to X and the asymptotic formulas still hold as long as α is bounded away from 0 and 1 by $(\log X)^{-c}$ for any $c > 0$. This is essentially equivalent to the normal order of $\pi_E(\alpha, \beta; X)$ being $(c_k(\beta) - c_k(\alpha))\pi_k(X)$ provided $\beta - \alpha > (\log X)^{-c}$ for any $c > 0$. The $(\log X)^{-c}$ can be further improved to $\exp(-c_1 \sqrt{\log X})$ for some $c_1 > 0$ if, instead of directly using the approximation (2.6) on major arcs, one separately discusses the cases for those q and $\chi(\text{mod } q)$ with $L(s, \chi)$ having a possible Siegel zero. However, this will not be the focus of this paper.

The organization of this paper is as follows. We first employ the Hardy-Littlewood method in section 2 to derive estimates on the number of representations of certain integers n as $n = r^2 - 4p$.

We then derive an estimate for a weighted average of special values of truncated Dirichlet L -functions in section 3. We employ these estimates to prove the main theorem in section 4. In section 5 we prove Theorem 2. Finally, we make some closing remarks in section 6.

2. A PROBLEM OF REPRESENTATIONS

In this section we employ the Hardy-Littlewood method (see [10]) in order to give an asymptotic formula for the number of representations of a negative integer $n \equiv 0$ or $1 \pmod{4}$ as

$$n = r^2 - 4p,$$

subject to $p \leq X$ being a prime, and $r \in \mathcal{K} \cap [0, 2\alpha\sqrt{p}]$, where $0 < \alpha < 1$. We give an asymptotic main term for the number of representations which is large when some local conditions are satisfied, and an error term which is small on average.

For convenience, we shall consider the following weighted number of representations.

$$R(n) = \sum_{\substack{p \leq X \\ r \leq (2\alpha\sqrt{p})^{\frac{1}{k}} \\ r^{2k} - 4p = n}} \log p. \quad (2.1)$$

Let $P = (\log X)^{2^{2k+3}}$. For a positive integer m , let $\text{ord}_2(m)$ be the non-negative integer such that $2^{\text{ord}_2(m)} \parallel m$, and we write $[m]_o = 2^{-\text{ord}_2(m)}m$ for the odd part of m .

Theorem 3. *For the $R(n)$ defined by (2.1), we have*

$$R(n) = \mathfrak{S}(n, P)J(n) + E(n) + O(X^{\frac{1}{2k}}(\log X)^{-3}), \quad (2.2)$$

where

$$\mathfrak{S}(n, P) = \sum_{m \leq P} \frac{\mu([m]_o)}{\phi([m]_o)} F(2^{\text{ord}_2(m)}, n) \sum_{\substack{\chi \pmod{[m]_o} \\ \chi^{2k} = \chi^0}}^* \chi(n),$$

the $*$ means that the summation is over the primitive characters modulo $[m]_o$, $F(2^{\text{ord}_2(m)}, n)$ is given by (2.13) below,

$$J(n) = \sum_{\substack{m \leq X \\ s \leq 4\alpha^2 m \\ s - 4m = n}} \frac{1}{2k} s^{\frac{1}{2k} - 1}$$

and $E(n)$ satisfies

$$\sum_{-n \leq 4X} |E(n)|^2 \ll \frac{X^{1+\frac{1}{k}}}{(\log X)^{20}}. \quad (2.3)$$

In the following, we can suppose $X(\log X)^{-2k+2} < -n \leq 4X$, since it follows from considering the ranges for p and r that $R(n)$ is trivially bounded by the error term $O(X^{\frac{1}{2k}}(\log X)^{-3})$ in (2.2) when $-n \leq X(\log X)^{-2k+2}$.

Let $g = 1 + (\log X)^{-5}$. To remove the dependence of r on p , we split the range of p into subintervals $(Xg^{-(l+1)}, Xg^{-l}]$, $l = 0, 1, 2, \dots, L$, where $L = \left\lceil \frac{\log \left(\frac{4X}{-n} \right)}{\log g} \right\rceil \ll (\log X)^5 \log \log X$. Then we have

$$R(n) = \sum_{l=0}^L R_l(n), \quad \text{where} \quad R_l(n) = \sum_{\substack{Xg^{-(l+1)} < p \leq Xg^{-l} \\ r \leq (2\alpha\sqrt{p})^{\frac{1}{k}} \\ r^{2k} - 4p = n}} \log p. \quad (2.4)$$

Let

$$R_l^*(n) = \sum_{\substack{Xg^{-(l+1)} < p \leq Xg^{-l} \\ r \leq (2\alpha\sqrt{Xg^{-l}})^{\frac{1}{k}} \\ r^{2k} - 4p = n}} \log p,$$

then it is clear that

$$0 \leq R_l^*(n) - R_l(n) \leq \sum_{\substack{Xg^{-(l+1)} < p \leq Xg^{-l} \\ (2\alpha\sqrt{p})^{\frac{1}{k}} < r \leq (2\alpha\sqrt{Xg^{-l}})^{\frac{1}{k}} \\ r^{2k} - 4p = n}} \log p \ll \log X + \left(\frac{X}{g^l} \right)^{\frac{1}{2k}} (\log X)^{-4}.$$

Note that, from the ranges of p and r in the above sum (and from the fact that $\alpha < 1$), it follows that there are at most $O_\alpha(1)$ values of l such that $R_l^*(n) - R_l(n)$ is non-zero, thus we have

$$R(n) = \sum_{l=0}^L R_l^*(n) + O(X^{\frac{1}{2k}}(\log X)^{-3}). \quad (2.5)$$

For a real number β , let

$$s_l(\beta) = \sum_{Xg^{-(l+1)} < p \leq Xg^{-l}} e(p\beta) \log p \quad \text{and} \quad t_l(\beta) = \sum_{r \leq (2\alpha\sqrt{Xg^{-l}})^{\frac{1}{k}}} e(r^{2k}\beta).$$

We first note that

$$R_l^*(n) = \int_{PX^{-1}}^{1+PX^{-1}} t_l(\beta) s_l(-4\beta) e(-n\beta) d\beta.$$

We divide the unit interval $[PX^{-1}, 1 + PX^{-1}]$ into two parts: the major arcs

$$\mathfrak{M} = \bigcup_{q \leq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^q \mathfrak{M}(q, a), \quad \text{where} \quad \mathfrak{M}(q, a) = \{\beta : |\beta - a/q| \leq PX^{-1}\},$$

and the minor arcs $\mathfrak{m} = [PX^{-1}, 1 + PX^{-1}] \setminus \mathfrak{M}$. It is clear that, for our choice of P , the $\mathfrak{M}(q, a)$'s are disjoint.

Note that for $l \leq L$, we have $X(\log X)^{-2k+2} \ll Xg^{-(l+1)} \ll Xg^{-l} \ll X$. Thus from [10] (Lemma 3.1), we see that, there exists a positive constant C such that whenever $1 \leq a \leq q \leq P$, $(a, q) = 1$, $\beta \in \mathfrak{M}(q, a)$, we have

$$s_l(-4\beta) = \frac{\mu(q/(4, q))}{\phi(q/(4, q))} u_l(-4(\beta - a/q)) + O(X \exp(-C\sqrt{\log X})), \quad (2.6)$$

where

$$u_l(\lambda) = \sum_{Xg^{-(l+1)} < m \leq Xg^{-l}} e(m\lambda).$$

For $\beta \in \mathfrak{M}(q, a)$, from Theorem 4.1 of [10], we trivially have

$$t_l(\beta) = q^{-1} S_{2k}(q, a) v_l(\beta - a/q) + O(P^{\frac{2}{3}}), \quad (2.7)$$

where

$$S_{2k}(q, a) = \sum_{m=1}^q e(am^{2k}/q) \quad \text{and} \quad v_l(\lambda) = \sum_{s \leq 4\alpha^2 X/g^l} \frac{1}{2k} s^{\frac{1}{2k}-1} e(s\lambda).$$

From [10, Lemma 2.8] we have

$$u_l(\lambda) \ll \min\{Xg^{-l}, \|\lambda\|^{-1}\}, \quad v_l(\lambda) \ll \min\{(Xg^{-l})^{\frac{1}{2k}}, \|\lambda\|^{-\frac{1}{2k}}\}. \quad (2.8)$$

Thus from (2.6), (2.7) and (2.8), we get

$$\begin{aligned} R_l^*(n) &= \sum_{q \leq P} \frac{\mu(q/(4, q))}{q\phi(q/(4, q))} \sum_{\substack{a=1 \\ (a,q)=1}}^q S_{2k}(q, a) e(-an/q) \int_{-P/X}^{P/X} u_l(-4\lambda) v_l(\lambda) e(-n\lambda) d\lambda + E_l(n) \\ &\quad + O\left(X^{\frac{1}{2k}} \exp\left(\frac{-C}{2}\sqrt{\log X}\right)\right), \end{aligned} \quad (2.9)$$

where

$$E_l(n) = \int_{\mathfrak{m}} s_l(-4\beta) t_l(\beta) e(-n\beta) d\beta.$$

Let

$$\mathfrak{S}(n, P) = \sum_{q \leq P} F(q, n) \quad \text{where} \quad F(q, n) = \frac{\mu(q/(4, q))}{q\phi(q/(4, q))} \sum_{\substack{a=1 \\ (a,q)=1}}^q S_{2k}(q, a) e(-an/q). \quad (2.10)$$

We note that $F(q, n)$ is multiplicative. (While it is well-known that $F(q, n)$ is multiplicative if the $(4, q)$ is not present, it is straightforward to check that the presence of $(4, q)$ does not affect the multiplicativity.)

For an odd prime p , we see that $F(p^h, n) = 0$ if $p \mid n$ or $h \geq 2$. If $p \nmid 2n$, we have

$$\begin{aligned} F(p, n) &= \frac{-1}{p(p-1)} \sum_{a=1}^{p-1} e(-an/p) \sum_{m=1}^p e(am^{2k}/p) \\ &= \frac{-1}{p(p-1)} \left[p \left(\sum_{\substack{m=1 \\ m^{2k} \equiv n \pmod{p}}}^p 1 \right) - p \right] = \frac{-1}{p-1} \sum_{\substack{\chi \pmod{p} \\ \chi^{2k} = \chi^0 \\ \chi \neq \chi^0}} \chi(n). \end{aligned} \quad (2.11)$$

Thus

$$F(q, n) = F(2^{\text{ord}_2(q)}, n) \cdot \frac{\mu([q]_o)}{\phi([q]_o)} \sum_{\substack{\chi \pmod{[q]_o} \\ \chi^{2k} = \chi^0}}^* \chi(n). \quad (2.12)$$

It is easy to check that

$$F(2^h, n) = \begin{cases} 0 & \text{if } h = 1 \text{ or } h \geq 4, \\ 1 & \text{if } h = 0 \text{ or } h = 2, \\ -2\chi_n(2) & \text{if } h = 3 \text{ and } n \equiv 1 \pmod{4}, \\ 0 & \text{if } h = 3, \quad k = 1 \text{ and } n \equiv 0 \pmod{4}, \\ -2 & \text{if } h = 3, \quad k > 1 \text{ and } n \equiv 0 \pmod{4}. \end{cases} \quad (2.13)$$

From the estimates (2.8), we now have

$$\left(\int_{-\frac{1}{2}}^{-P/X} + \int_{P/X}^{\frac{1}{2}} \right) u_l(-4\lambda) v_l(\lambda) e(-n\lambda) d\lambda \ll \int_{P/X}^{\frac{1}{2}} \lambda^{-1-\frac{1}{2k}} d\lambda \ll \left(\frac{X}{P} \right)^{\frac{1}{2k}}.$$

From this and (2.9), and the fact that $\mathfrak{S}(n, P) \ll (\log X)^{k-1}$, we have

$$R_l^*(n) = \mathfrak{S}(n, P) \int_{-\frac{1}{2}}^{\frac{1}{2}} u_l(-4\lambda) v_l(\lambda) e(-n\lambda) d\lambda + E_l(n) + O(X^{\frac{1}{2k}} (\log X)^{k-1} P^{-\frac{1}{2k}}), \quad (2.14)$$

where it is clear that $X^{\frac{1}{2k}} (\log X)^{k-1} P^{-\frac{1}{2k}} \ll X^{\frac{1}{2k}} (\log X)^{-16}$.

Let

$$J_l^*(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} u_l(-4\lambda) v_l(\lambda) e(-n\lambda) d\lambda.$$

Then it is obvious that

$$J_l^*(n) = \sum_{\substack{Xg^{-(l+1)} < m \leq Xg^{-l} \\ s \leq 4\alpha^2 Xg^{-l} \\ s-4m=n}} \frac{1}{2k} s^{\frac{1}{2k}-1}.$$

Similar to (2.5), we have

$$\sum_{l=0}^L J_l^*(n) = J(n) + O\left(X^{\frac{1}{2k}} (\log X)^{-4}\right).$$

Let

$$E(n) = \sum_{l=0}^L E_l(n),$$

then we have proved (2.2). To prove (2.3), we first recall $L \leq \log^6 X$ and observe that

$$\sum_n |E(n)|^2 \ll (\log X)^{12} \max_{0 \leq l \leq L} \sum_{-n \leq 4X} |E_l(n)|^2. \quad (2.15)$$

By Bessel's inequality, we have

$$\sum_{-n \leq 4X} |E_l(n)|^2 \ll \int_{\mathfrak{m}} |s_l(4\beta) t_l(\beta)|^2 d\beta. \quad (2.16)$$

By Weyl's inequality, we have

$$t_l(\beta) \ll X^{\frac{1}{2k}} P^{-2^{1-2k}} \log X \ll X^{\frac{1}{2k}} (\log X)^{-15}, \quad \beta \in \mathfrak{m}.$$

Thus

$$\int_{\mathfrak{m}} |s_l(4\beta) t_l(\beta)|^2 d\beta \ll X^{\frac{1}{k}} (\log X)^{-30} \int_0^1 |s_l(\beta)|^2 d\beta \ll X^{\frac{1}{k}} (\log X)^{-28} \left(\frac{X}{g^l} - \frac{X}{g^{l+1}} \right) \ll \frac{X^{1+\frac{1}{k}}}{(\log X)^{33}},$$

which, along with (2.15) and (2.16), implies (2.3).

3. A WEIGHTED SUM

In this section we derive an estimate for a weighted sum of special values of truncated Dirichlet L -functions. More precisely, letting $L_0(d) = \sum_{n \leq X^{2/3}} \frac{\chi_d(n)}{n}$ and

$$K_0(X) = \frac{1}{2\pi} \sum_{p \leq X} \log p \sum_{0 < r \leq (2\alpha\sqrt{p})^{\frac{1}{k}}} \sum_{\substack{r^{2k} - 4p = df^2 \\ f \leq (\log X)^2 \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} L_0(d), \quad (3.1)$$

we prove

Theorem 4. *We have*

$$K_0(X) = \left(\frac{2k}{3k+1} c_k(\alpha) + O((\log X)^{-2}) \right) X^{\frac{3}{2} + \frac{1}{2k}}. \quad (3.2)$$

We first note that, from Theorem 3, we have

$$K_0(X) = \frac{1}{2\pi} \sum_{\substack{f \leq (\log X)^2 \\ 0 < -df^2 \leq 4X \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} L_0(d) \left(\mathfrak{S}(df^2, P) J(df^2) + E(df^2) + O(X^{\frac{1}{2k}} (\log X)^{-3}) \right), \quad (3.3)$$

We note that

$$\begin{aligned} \sum_{f,d} \sqrt{|d|} |L_0(d) E(df^2)| &\ll \log X \sqrt{\sum_{-n \leq 4X} |n| \tau^2(|n|)} \sqrt{\sum_{-n \leq 4X} |E(n)|^2} \\ &\ll X (\log X)^{\frac{5}{2}} \left(\frac{X^{1+\frac{1}{k}}}{(\log X)^{20}} \right)^{\frac{1}{2}} \ll \frac{X^{\frac{3}{2}+\frac{1}{2k}}}{(\log X)^7}, \end{aligned} \quad (3.4)$$

also the contribution of the term $O(X^{\frac{1}{2k}} (\log X)^{-3})$ in the sum of (3.3) contributes $O(X^{\frac{3}{2}+\frac{1}{2k}} (\log X)^{-2})$.

Thus, we have

$$K_0(X) = \frac{1}{2\pi} \sum_{\substack{f \leq (\log X)^2 \\ 0 < -df^2 \leq 4X \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} L_0(d) \mathfrak{S}(df^2, P) J(df^2) + O(X^{\frac{3}{2}+\frac{1}{2k}} (\log X)^{-2}). \quad (3.5)$$

Now, if $-df^2 \leq 4(1 - \alpha^2)X$, then for each $s \leq \frac{-\alpha^2 df^2}{1 - \alpha^2}$ satisfying $s \equiv df^2 \pmod{4}$ we have a unique $m = \frac{s - df^2}{4} \in [\frac{s}{4\alpha^2}, X]$. If $4(1 - \alpha^2)X < -df^2 \leq 4X$, then each $s \leq 4X + df^2$ congruent to df^2 modulo 4 gives a unique $m = \frac{s - df^2}{4} \in [\frac{s}{4\alpha^2}, X]$. Note that for any $N > 0$, $h \in \mathbb{Z}$, we have

$$\sum_{\substack{s \leq N \\ s \equiv h \pmod{4}}} \frac{1}{2k} s^{\frac{1}{2k}-1} = \frac{1}{8k} \sum_{s \leq N} s^{\frac{1}{2k}-1} + O(1) = \frac{1}{4} N^{\frac{1}{2k}} + O(1).$$

Thus,

$$J(df^2) = \begin{cases} \frac{1}{4} \left(\frac{-\alpha^2 df^2}{1 - \alpha^2} \right)^{\frac{1}{2k}} + O(1) & \text{if } -df^2 \leq 4(1 - \alpha^2)X, \\ \frac{1}{4} (4X + df^2)^{\frac{1}{2k}} + O(1) & \text{if } 4(1 - \alpha^2)X < -df^2 \leq 4X. \end{cases} \quad (3.6)$$

Let

$$K_0^a(X) = \frac{1}{8\pi} \left(\frac{\alpha^2}{1 - \alpha^2} \right)^{\frac{1}{2k}} \sum_{\substack{f \leq (\log X)^2 \\ 0 < -df^2 \leq 4(1 - \alpha^2)X \\ d \equiv 0,1 \pmod{4}}} |d|^{\frac{1}{2}+\frac{1}{2k}} f^{\frac{1}{k}} \sum_{n \leq X^{2/3}} \frac{\chi_d(n)}{n} \sum_{2^j m \leq P} \frac{\mu(m)}{\phi(m)} F(2^j, df^2) \sum_{\substack{\chi \pmod{m} \\ \chi^{2k} = \chi^0}}^* \chi(df^2)$$

and

$$K_0^b(X) = \frac{1}{8\pi} \sum_{\substack{f \leq (\log X)^2 \\ 4(1-\alpha^2)X < -df^2 \leq 4X \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} (4X + df^2)^{\frac{1}{2k}} \sum_{n \leq X^{2/3}} \frac{\chi_d(n)}{n} \sum_{2^j m \leq P} \frac{\mu(m)}{\phi(m)} F(2^j, df^2) \sum_{\substack{\chi \pmod{m} \\ \chi^{2k} = \chi^0}}^* \chi(df^2),$$

where in both expressions, the second to last summation is subject to m being odd, $j = 0, 2, 3$ and $F(2^j, df^2)$ being given by (2.13). Then we have

$$K_0(X) = K_0^a(X) + K_0^b(X) + O(X^{\frac{3}{2} + \frac{1}{2k}} (\log X)^{-2}). \quad (3.7)$$

Note that

$$\frac{2^{1+\frac{1}{k}} k \alpha^{\frac{1}{k}} (1-\alpha^2)^{\frac{3}{2}}}{(3k+1)\pi} + \frac{2^{\frac{1}{k}}}{\pi} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt = \frac{2^{1+\frac{1}{k}}}{(3k+1)\pi} \int_0^{\alpha} t^{\frac{1}{k}-1} \sqrt{1-t^2} dt.$$

(Both sides are equal to 0 at $\alpha = 0$ and have the same derivatives with respect to α .) Then Theorem 4 follows from the following lemma.

Lemma 3.1. *We have*

$$K_0^a(X) = \left(\frac{1}{3} + \frac{2}{3} \delta(k) \right) \frac{2^{1+\frac{1}{k}} k \alpha^{\frac{1}{k}} (1-\alpha^2)^{\frac{3}{2}}}{(3k+1)\pi} X^{\frac{3}{2} + \frac{1}{2k}} + O\left(\frac{X^{\frac{3}{2} + \frac{1}{2k}}}{(\log X)^4} \right), \quad (3.8)$$

and

$$K_0^b(X) = \left(\frac{1}{3} + \frac{2}{3} \delta(k) \right) \frac{2^{\frac{1}{k}} X^{\frac{3}{2} + \frac{1}{2k}}}{\pi} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{X^{\frac{3}{2} + \frac{1}{2k}}}{(\log X)^4} \right). \quad (3.9)$$

The proofs of (3.8) and (3.9) are almost identical. In view of this, we will only give a proof for (3.9). We first note that

$$K_0^b(X) = \frac{1}{8\pi} \sum_{j,f,m,n} \frac{\mu(m)}{n\phi(m)} \sum_{\substack{\chi \pmod{m} \\ \chi^{2k} = \chi^0}}^* \sum_{\substack{\frac{4(1-\alpha^2)X}{f^2} < -d \leq \frac{4X}{f^2} \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} (4X + df^2)^{\frac{1}{2k}} \chi_d(n) F(2^j, df^2) \chi(df^2),$$

where the summation over j, f, m, n is also subject to $f \leq (\log X)^2$, $2^j m \leq P = (\log X)^{2k+3}$, $j = 0, 2, 3$, $n \leq X^{2/3}$.

We split the sum into two parts: K_0^{be} subject to $d \equiv 0 \pmod{4}$ and K_0^{bo} subject to $d \equiv 1 \pmod{4}$. Write $d = 4D$ in K_0^{be} , then we have

$$K_0^{be} = \frac{2^{\frac{1}{k}}}{4\pi} \sum_{\substack{j,f,m,n \\ 2 \nmid mn}} \frac{\mu(m) F(2^j, 4)}{n\phi(m)} \sum_{\substack{\chi \pmod{m} \\ \chi^{2k} = \chi^0}}^* \chi(4f^2) \sum_{\substack{\frac{(1-\alpha^2)X}{f^2} < -D \leq \frac{X}{f^2}}} \sqrt{|D|} (X + Df^2)^{\frac{1}{2k}} \chi_D(n) \chi(D),$$

where we have replaced $F(2^j, 4Df^2)$ by $F(2^j, 4)$ in view of (2.13). If $\psi = \chi(\frac{\cdot}{n}) \neq \chi^0$, then from the Pólya-Vinogradov estimate, the innermost sum of K_0^{be} is

$$\begin{aligned} &= \psi(-1) \int_{\frac{(1-\alpha^2)X}{f^2}}^{\frac{X}{f^2}} \sqrt{t}(X - f^2t)^{\frac{1}{2k}} \cdot \frac{d}{dt} \left[\sum_{s \leq t} \psi(s) \right] dt \\ &\ll \frac{X^{\frac{1}{2} + \frac{1}{2k}} \sqrt{mn} \log(mn+1)}{f} \ll \frac{X^{\frac{5}{6} + \frac{1}{2k}} P}{f}. \end{aligned} \quad (3.10)$$

It is then easy to see that the terms of K_0^{be} with $\chi(\frac{\cdot}{n}) \neq \chi^0$ contribute at most $O(X^{1+\frac{1}{2k}})$. Now, we notice that $\chi(\frac{\cdot}{n}) = \chi^0$ only when $\chi = (\frac{\cdot}{m})$ (since $\chi \pmod{m}$ is primitive) and $n = mw^2$ for some $w \in \mathbb{N}$. Thus, we have

$$K_0^{be} = \frac{2^{\frac{1}{k}}}{4\pi} \sum_{\substack{j,f,m,w \\ (m,f)=1 \\ 2 \nmid mw}} \frac{\mu(m)F(2^j, 4)}{m\phi(m)w^2} \sum_{\substack{\frac{(1-\alpha^2)X}{f^2} < -D \leq \frac{X}{f^2} \\ (D,mw)=1}} \sqrt{|D|}(X + Df^2)^{\frac{1}{2k}} + O(X^{1+\frac{1}{2k}}), \quad (3.11)$$

where the summation over j, f, m, w is also subject to $f \leq (\log X)^2$, $2^j m \leq P = (\log X)^{2^{2k+3}}$, $j = 0, 2$ or 3 , and $w \leq \frac{X^{\frac{1}{3}}}{\sqrt{m}}$. Note that the inner sum is

$$\begin{aligned} \int_{\frac{(1-\alpha^2)X}{f^2}}^{\frac{X}{f^2}} \sqrt{t}(X - f^2t)^{\frac{1}{2k}} d \sum_{\substack{u \leq t \\ (u,mw)=1}} 1 &= \int_{\frac{(1-\alpha^2)X}{f^2}}^{\frac{X}{f^2}} \sqrt{t}(X - f^2t)^{\frac{1}{2k}} d \left\{ \frac{\phi(mw)}{mw} t + O(mw) \right\} \\ &= \frac{\phi(mw)X^{\frac{3}{2} + \frac{1}{2k}}}{mf^3} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{mwX^{\frac{1}{2} + \frac{1}{2k}}}{f}\right) \end{aligned} \quad (3.12)$$

From this and (3.11), we have

$$K_0^{be} = \left(\frac{2^{\frac{1}{k}} X^{\frac{3}{2} + \frac{1}{2k}}}{4\pi} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt \right) \sum_{\substack{j,f,m,w \\ (m,f)=1 \\ 2 \nmid mw}} \frac{\mu(m)\phi(mw)F(2^j, 4)}{m^2\phi(m)w^3f^3} + O(X^{1+\frac{1}{2k}}). \quad (3.13)$$

Now we note that the sum over j, f, m, w is equal to

$$\begin{aligned} &\sum_{\substack{2^j m \leq P \\ 2 \nmid m}} \frac{\mu(m)F(2^j, 4)}{m^2\phi(m)} \left(\sum_{\substack{f=1 \\ (f,m)=1}}^{\infty} \frac{1}{f^3} + O\left(\frac{1}{(\log X)^4}\right) \right) \left(\sum_{\substack{w=1 \\ 2 \nmid w}}^{\infty} \frac{\phi(mw)}{w^3} + O\left(\frac{m^{\frac{3}{2}}}{X^{1/3}}\right) \right) \\ &= \sum_{\substack{2^j m \leq P \\ 2 \nmid m}} \frac{\mu(m)F(2^j, 4)}{m^2\phi(m)} \sum_{\substack{f=1 \\ (f,m)=1}}^{\infty} \frac{1}{f^3} \sum_{\substack{w=1 \\ 2 \nmid w}}^{\infty} \frac{\phi(mw)}{w^3} + O\left(\frac{1}{(\log X)^4}\right) \\ &= \frac{16}{7} \delta(k) \sum_{\substack{f,m,w \\ 2 \nmid mw \\ (f,m)=1}} \frac{\mu(m)\phi(mw)}{m^2\phi(m)w^3f^3} + O\left(\frac{1}{(\log X)^4}\right), \end{aligned} \quad (3.14)$$

where in the last sum, all variables f, m, k are running over all odd positive integers. (Here we have factored out the powers of 2 in f , which yields the constant factor $1 + 2^{-3} + 2^{-6} + \dots = \frac{8}{7}$), and it is clear to see that a factor $2\delta(k)$ arises from summing up $F(2^0, 4)$, $F(2^2, 4)$ and $F(2^3, 4)$ according to (2.13).

Lemma 3.2. *We have*

$$C := \sum_{\substack{f, m, w \\ 2 \nmid mw \\ (f, m) = 1}} \frac{\mu(m)\phi(mw)}{m^2\phi(m)w^3f^3} = 1.$$

Proof. We first note that

$$\begin{aligned} C &= \sum_{\substack{m, w \\ 2 \nmid mw}} \frac{\mu(m)\phi(mw)}{m^2\phi(m)w^3} \zeta(3) \prod_{l|2m} \left(1 - \frac{1}{l^3}\right) \\ &= \frac{7}{8} \zeta(3) \sum_{\substack{m, w \\ 2 \nmid mw}} \frac{\mu(m)\phi(mw)}{m^2\phi(m)w^3} \sum_{v|m} \frac{\mu(v)}{v^3} \\ &= \frac{7}{8} \zeta(3) \sum_{\substack{K=1 \\ 2 \nmid K}}^{\infty} \frac{\phi(K)a(K)}{K^3}, \end{aligned}$$

where

$$a(K) = \sum_{m|K} \frac{\mu(m)m}{\phi(m)} \sum_{v|m} \frac{\mu(v)}{v^3}$$

which is obviously a multiplicative function. Note that, for a prime l , $a(l) = a(l^2) = a(l^3) = \dots$. Thus

$$\begin{aligned} C &= \frac{7}{8} \zeta(3) \prod_{l>2} \left(1 + a(l) \left(\frac{\phi(l)}{l^3} + \frac{\phi(l^2)}{l^6} + \dots\right)\right) \\ &= \frac{7}{8} \zeta(3) \prod_{l>2} \left(1 + \frac{a(l)}{l(l+1)}\right). \end{aligned} \tag{3.15}$$

It is easy to see that

$$a(l) = 1 + \frac{-l}{l-1} \left(1 - \frac{1}{l^3}\right) = -\frac{l+1}{l^2}.$$

Thus, from this and (3.15), we have

$$C = \frac{7}{8} \zeta(3) \prod_{l>2} \left(1 - \frac{1}{l^3}\right) = 1,$$

which proves the lemma. \square

From (3.13), (3.14) and Lemma 3.2, we have

$$K_0^{be} = \frac{2^{2+\frac{1}{k}} X^{\frac{3}{2}+\frac{1}{2k}}}{7\pi} \delta(k) \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{X^{\frac{3}{2}+\frac{1}{2k}}}{(\log X)^4}\right). \quad (3.16)$$

Now we try to find an asymptotic formula for K_0^{bo} . We first note that

$$K_0^{bo} = \frac{1}{8\pi} \sum_{j,f,m,n} \frac{\mu(m)}{n\phi(m)} \sum_{\substack{\chi(\bmod m) \\ \chi^{2k}=\chi^0}}^* \chi(f^2) \sum_{\substack{\frac{4(1-\alpha^2)X}{f^2} < -d \leq \frac{4X}{f^2} \\ d \equiv 1 \pmod{4}}} \sqrt{|d|} (4X + df^2)^{\frac{1}{2k}} F(2^j, df^2) \chi_d(n) \chi(d).$$

We replace the restriction $d \equiv 1 \pmod{4}$ by introducing a characteristic function $\frac{1}{2}(\chi_4(d) + \chi_{-4}(d))$.

When $j = 0$ or 2 , we have $F(2^j, df^2) = 1$. We note that $\chi_{-4}(d)\chi_d(n)\chi(d)$ and, if $\chi(\frac{\cdot}{n}) \neq \chi^0$, $\chi_4(d)\chi_d(n)\chi(d)$ are both non-principal characters modulo $4mn$. Thus, with the same estimate as (3.10), we see that the total contribution from the terms containing $\chi_{-4}(d)$ and the terms containing $\chi_4(d)$ but with $\chi(\frac{\cdot}{n}) \neq \chi^0$ is $O(X^{1+\frac{1}{2k}})$. We remark that $\chi(\frac{\cdot}{n}) = \chi^0$ if and only if $\chi = (\frac{\cdot}{m})$ and $n = mw^2$ for some $w \in \mathbb{N}$.

When $j = 3$ and f is odd, we have $F(2^j, df^2) = -2(\frac{2}{d})$. Note then

$$\frac{1}{2}(\chi_4(d) + \chi_{-4}(d))\chi_d(n)\chi(d)F(2^3, df^2) = -\left(\left(\frac{2}{d}\right) + \left(\frac{-2}{d}\right)\right)\chi(d)\left(\frac{d}{n}\right). \quad (3.17)$$

While the terms associated with non-principal characters contribute at most $O(X^{1+\frac{1}{2k}})$ from the Pólya-Vinogradov estimate, the only possible principal characters arise from $\chi = (\frac{\cdot}{m})$. When $\chi = (\frac{\cdot}{m})$, apart from the factor -1 , (3.17) is the sum of two non-principal characters modulo $8mn$ if mn is not twice a square. These terms again give a contribution $O(X^{1+\frac{1}{2k}})$ to K_0^{bo} with the same estimate as (3.10). When $n = 2mw^2$ for some integer w , (3.17) is simply equal to $-(1 + \chi_{-4}(d))$. The terms containing $\chi_{-4}(d)$ again contribute at most $O(X^{1+\frac{1}{2k}})$ to K_0^{bo} .

When $j = 3$ and f is even, we have $F(2^j, df^2) = -2(1 - \delta(k))$. Thus

$$\frac{1}{2}(\chi_4(d) + \chi_{-4}(d))\chi_d(n)\chi(d)F(2^3, df^2) = -(1 - \delta(k))\left(\left(\frac{4}{d}\right) + \left(\frac{-4}{d}\right)\right)\chi(d)\left(\frac{d}{n}\right). \quad (3.18)$$

It is clear that the only principal character arises from $\chi = (\frac{\cdot}{m})$ and $n = mw^2$ for some $w \in \mathbb{N}$, in which case the term in (3.18) contributing to the main term is equal to $-(1 - \delta(k))$.

From the above discussion, we have

$$\begin{aligned}
K_0^{bo} &= \frac{1}{16\pi} \sum_{\substack{j,f,m,w \\ j=0,2 \\ (m,2f)=1}} \frac{\mu(m)}{m\phi(m)w^2} \sum_{\substack{\frac{4(1-\alpha^2)X}{f^2} < -d \leq \frac{4X}{f^2} \\ (d,2mw)=1}} \sqrt{|d|}(4X + df^2)^{\frac{1}{2k}} - \\
&\quad - \frac{1}{16\pi} \sum_{\substack{f,m,w \\ (m,f)=1 \\ 2|m f}} \frac{\mu(m)}{m\phi(m)w^2} \sum_{\substack{\frac{4(1-\alpha^2)X}{f^2} < -d \leq \frac{4X}{f^2} \\ (d,2mw)=1}} \sqrt{|d|}(4X + df^2)^{\frac{1}{2k}} - \\
&\quad - \frac{1-\delta(k)}{8\pi} \sum_{\substack{f,m,w \\ (m,2f)=1 \\ 2|f}} \frac{\mu(m)}{m\phi(m)w^2} \sum_{\substack{\frac{4(1-\alpha^2)X}{f^2} < -d \leq \frac{4X}{f^2} \\ (d,2mw)=1}} \sqrt{|d|}(4X + df^2)^{\frac{1}{2k}} \\
&\quad + O(X^{1+\frac{1}{2k}}), \tag{3.19}
\end{aligned}$$

where in the first sum the summation over f, m, w is also subject to $f \leq (\log X)^2$, $2^j m \leq P$ and $w \leq \frac{X^{\frac{1}{3}}}{\sqrt{2^j m}}$, and in the second and the third sums the summation over f, m, w is also subject to $f \leq (\log X)^2$, $m \leq \frac{P}{8}$, $w \leq \frac{X^{\frac{1}{3}}}{\sqrt{2m}}$ and $w \leq \frac{X^{\frac{1}{3}}}{\sqrt{m}}$, respectively. Similar to (3.12), we see that the innermost sums of the three sums in (3.19) are equal to

$$\frac{2^{2+\frac{1}{k}}\phi(2mw)X^{\frac{3}{2}+\frac{1}{2k}}}{mwf^3} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{mwX^{\frac{1}{2}+\frac{1}{2k}}}{f}\right). \tag{3.20}$$

From (3.19) and (3.20), and by an argument similar to (3.14), we get

$$K_0^{bo} = \frac{2^{\frac{1}{k}} c^{bo} X^{\frac{3}{2}+\frac{1}{2k}}}{4\pi} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{X^{\frac{3}{2}+\frac{1}{2k}}}{(\log X)^4}\right), \tag{3.21}$$

where

$$c^{bo} = 2 \sum_{\substack{f,m,w=1 \\ (m,2f)=1}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3} - \sum_{\substack{f,m,w=1 \\ (m,f)=1 \\ 2|m f}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3} - 2(1-\delta(k)) \sum_{\substack{f,m,w=1 \\ (m,2f)=1 \\ 2|f}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3}.$$

From Lemma 3.2, we see that

$$\begin{aligned}
\sum_{\substack{f,m,w=1 \\ (m,2f)=1}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3} &= \frac{8}{7} \sum_{\substack{2|m f \\ (m,f)=1}} \frac{\mu(m)}{m^2\phi(m)f^3} \sum_{2|h} \frac{\phi(mh)}{h^3} \left(1 + \frac{2}{2^3} + \frac{4}{4^3} + \frac{8}{8^3} + \cdots\right) \\
&= \frac{32}{21} C = \frac{32}{21}.
\end{aligned}$$

Similarly, we have

$$\sum_{\substack{f, m, w=1 \\ (m, f)=1 \\ 2 \nmid mf}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3} = \frac{4}{3} \quad \text{and} \quad \sum_{\substack{f, m, w=1 \\ (m, 2f)=1 \\ 2 \nmid f}}^{\infty} \frac{\mu(m)\phi(2mw)}{m^2\phi(m)w^3f^3} = \frac{4}{21}.$$

Combining these with (3.21) have,

$$K_0^{bo} = \left(\frac{1}{3} + \frac{2}{21}\delta(k) \right) \frac{2^{\frac{1}{k}} X^{\frac{3}{2} + \frac{1}{2k}}}{\pi} \int_0^{\alpha^2} t^{\frac{1}{2k}} \sqrt{1-t} dt + O\left(\frac{X^{\frac{3}{2} + \frac{1}{2k}}}{(\log X)^4} \right). \quad (3.22)$$

From (3.16) and (3.22), we have proved (3.9).

4. PROOF OF THEOREM 1

By Deuring's theorem (see [2], [3]), we know that for a given prime p , and an integer $r \in (-2\sqrt{p}, 2\sqrt{p})$, $N(p, r)$, the number of elliptic curves $E(a, b) : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{F}_p$ and $a_p(E(a, b)) = r$, is given by

$$N(p, r) = \frac{pH(r^2 - 4p)}{2} + O(p), \quad (4.1)$$

where $H(r^2 - 4p)$ denotes the Kronecker class number which is given by

$$H(r^2 - 4p) = 2 \sum_{\substack{r^2 - 4p = df^2 \\ d \equiv 0, 1 \pmod{4}}} \frac{h(d)}{w(d)}. \quad (4.2)$$

Here $w(d)$ and $h(d)$ respectively denote the number of the units and class number of the order of discriminant d .

Proof of Theorem 1. Changing the order of summation in $S_\alpha(U, V, A, B; \mathcal{K}; X)$ and recalling (4.1), we get

$$\begin{aligned} S_\alpha(U, V, A, B; X) &= \frac{1}{AB} \sum_{p \leq X} \sum_{\substack{0 < r \leq 2\alpha\sqrt{p} \\ r \in \mathcal{K}}} \left(\frac{A}{p} + O(1) \right) \left(\frac{B}{p} + O(1) \right) N(p, r) \\ &= (1 + O((\log X)^{-1})) M(X) + O(X^{\frac{1}{2k}}) \end{aligned} \quad (4.3)$$

where

$$M(X) = \frac{1}{2} \sum_{\substack{p \leq X \\ 0 < r \leq 2\alpha\sqrt{p} \\ r \in \mathcal{K}}} \frac{H(r^2 - 4p)}{p}. \quad (4.4)$$

Theorem 1 now follows from our next result.

Theorem 5. For fixed $0 < \alpha < 1$, and sufficiently large X , we have

$$M(X) \sim c_k(\alpha) \pi_k(X).$$

Proof. Using (4.2), we have

$$M(X) = \sum_{p \leq X} \frac{1}{p} \sum_{0 < r \leq (2\alpha\sqrt{p})^{\frac{1}{k}}} \sum_{\substack{r^{2k} - 4p = df^2 \\ d \equiv 0,1 \pmod{4}}} \frac{h(d)}{w(d)}. \quad (4.5)$$

We note that in the sum $d < 0$, thus using Dirichlet's class number formula and well known estimates on L -functions, we have

$$\frac{h(d)}{w(d)} = \frac{\sqrt{|d|}}{2\pi} L(1, \chi_d) \ll \sqrt{|d|} \log(|d| + 1), \quad (4.6)$$

where χ_d is the Kronecker symbol. We see that the terms in the sum of (4.5) subject to $f > (\log X)^2$ contribute at most

$$\begin{aligned} & \ll \sum_{p \leq X} \frac{1}{p} \sum_{r \leq (2\alpha\sqrt{p})^{\frac{1}{k}}} \sum_{\substack{r^{2k} - 4p = df^2 \\ f > (\log X)^2}} \frac{\sqrt{p} \log p}{f} \\ & \ll \log X \sum_{(\log X)^2 < f \leq 2\sqrt{X}} \frac{1}{f} \sum_{r \leq (2\alpha\sqrt{X})^{\frac{1}{k}}} \sum_{\substack{\frac{r^{2k}}{4} < p \leq X \\ 4p \equiv r^2 \pmod{f^2}}} \frac{1}{\sqrt{p}} \\ & \ll \log X \sum_{(\log X)^2 < f \leq 2\sqrt{X}} \frac{1}{f} \sum_{r \leq (2\alpha\sqrt{X})^{\frac{1}{k}}} \left(\frac{\sqrt{X}}{f^2} + \frac{1}{r^{2k}} \right) \\ & \ll \frac{X^{\frac{1}{2} + \frac{1}{2k}}}{(\log X)^3}. \end{aligned} \quad (4.7)$$

This, together with (4.5), gives

$$M(X) = \sum_{p \leq X} \frac{1}{p} \sum_{0 < r \leq (2\alpha\sqrt{p})^{\frac{1}{k}}} \sum_{\substack{r^{2k} - 4p = df^2 \\ f \leq (\log X)^2 \\ d \equiv 0,1 \pmod{4}}} \frac{h(d)}{w(d)} + O\left(\frac{X^{\frac{1}{2} + \frac{1}{2k}}}{(\log X)^3}\right). \quad (4.8)$$

Since χ_d is non-principal, from the Pólya-Vinogradov Theorem, we have

$$\begin{aligned} L(1, \chi_d) &= \sum_{n \leq X^{2/3}} \frac{\chi_d(n)}{n} + O\left(\frac{\sqrt{|d|} \log |d|}{X^{2/3}}\right) \\ &= L_0(d) + O(|d|^{\frac{1}{2}} X^{-2/3} \log X), \quad \text{say.} \end{aligned} \quad (4.9)$$

Combining (4.6), (4.8) and (4.9) together, we get

$$M(X) = M_0(X) + O\left(\frac{X^{\frac{1}{2} + \frac{1}{2k}}}{(\log X)^3}\right), \quad (4.10)$$

where

$$M_0(X) = \frac{1}{2\pi} \sum_{p \leq X} \frac{1}{p} \sum_{r \leq (2\alpha\sqrt{p})^{\frac{1}{k}}} \sum_{\substack{r^{2k} - 4p = df^2 \\ f \leq (\log X)^2 \\ d \equiv 0,1 \pmod{4}}} \sqrt{|d|} L_0(d). \quad (4.11)$$

Using Theorem 4 and partial summation, we have

$$M_0(X) = \int_2^X \frac{1}{t \log t} dK_0(t) = c_k(\alpha) \pi_k(X) + O\left(\frac{X^{\frac{1}{2} + \frac{1}{2k}}}{(\log X)^3}\right).$$

Combining this with (4.10) proves Theorem 5.

5. PROOF OF THEOREM 2

From Theorem 1, it is clear that Theorem 2 follows from

$$F_\alpha(U, V, A, B; \mathcal{K}; X) := \frac{1}{AB} \sum_{\substack{U < a \leq U+A \\ V < b \leq V+B}} \pi_{E(a,b)}(\alpha; \mathcal{K}; X)^2 = (1 + o(1))(c_k(\alpha) \pi_k(X))^2. \quad (5.1)$$

Note that the left side of (5.1) is equal to

$$\frac{1}{AB} \sum_{\substack{U < a \leq U+A \\ V < b \leq V+B}} \sum_{\substack{p, q \leq X \\ a_{E(a,b)}(p) \leq 2\alpha\sqrt{p} \\ a_{E(a,b)}(q) \leq 2\alpha\sqrt{q} \\ a_{E(a,b)}(p), a_{E(a,b)}(q) \in \mathcal{K}}} 1. \quad (5.2)$$

From Theorem (1), the terms with $p = q$ in (5.2) contribute $O(\pi_k(X))$. Thus we have

$$F_\alpha(U, V, A, B; \mathcal{K}; X) = \frac{1}{AB} \sum_{p \neq q \leq X} \sum_{\substack{r \leq (2\alpha\sqrt{p})^{\frac{1}{k}} \\ s \leq (2\alpha\sqrt{q})^{\frac{1}{k}}}} \sum_{\substack{U < a \leq U+A \\ V < b \leq V+B \\ a_{E(a,b)}(p) = r^k \\ a_{E(a,b)}(q) = s^k}} 1 + O(\pi_k(X)). \quad (5.3)$$

From the Chinese Remainder Theorem, the inner sum of (5.3) is equal to

$$\left(\frac{A}{pq} + O(1)\right) \left(\frac{B}{pq} + O(1)\right) N(p, r^k) N(q, s^k). \quad (5.4)$$

When $A, B > (X \log X)^2$, we see that the main term $\frac{ABN(p, r^k)N(q, s^k)}{p^2 q^2}$ in (5.4) dominates the error term by at least a factor $(\log X)^2$. Thus, from (5.3) and (5.4), we have

$$\begin{aligned}
F_\alpha(U, V, A, B; \mathcal{K}; X) &= \left(1 + O\left(\frac{1}{(\log X)^2}\right)\right) \sum_{p \neq q \leq X} \sum_{\substack{r \leq (2\alpha\sqrt{p})^{\frac{1}{k}} \\ s \leq (2\alpha\sqrt{q})^{\frac{1}{k}}}} \frac{N(p, r^k)}{p^2} \frac{N(q, s^k)}{q^2} + O(\pi_k(X)) \\
&= \frac{1}{4} \left(1 + O\left(\frac{1}{(\log X)^2}\right)\right) \sum_{p \neq q \leq X} \sum_{\substack{r \leq (2\alpha\sqrt{p})^{\frac{1}{k}} \\ s \leq (2\alpha\sqrt{q})^{\frac{1}{k}}}} \frac{H(r^{2k} - 4p)}{p} \frac{H(s^{2k} - 4q)}{q} + O(\pi_k(X)) \\
&= \left(1 + O\left(\frac{1}{(\log X)^2}\right)\right) M(X)^2 + O(\pi_k(X)). \tag{5.5}
\end{aligned}$$

(2) then follows from Theorem 5.

6. FURTHER REMARKS

In [5], David and Pappalardi considered the average Frobenius distributions of elliptic curves. More precisely, they showed

Theorem 6. *Let r be an odd integer, $A, B \geq 1$. Let*

$$\pi_E^r(X) := \#\{p \leq X : a_E(p) = r\}$$

and

$$\pi_{1/2}(X) = \int_2^X \frac{dt}{2\sqrt{t} \log t} \sim \frac{\sqrt{X}}{\log X}.$$

For every $c > 0$, we have

$$\frac{1}{4AB} \sum_{\substack{|a| \leq A \\ |b| \leq B}} \pi_{E(a,b)}^r(X) = C_r \pi_{1/2}(X) + O\left(\left(\frac{1}{A} + \frac{1}{B}\right) X^{3/2} + \frac{X^{5/2}}{AB} + \frac{\sqrt{X}}{(\log X)^c}\right) \tag{6.1}$$

where

$$C_r = \frac{2}{\pi} \prod_{l|r} \frac{l^2}{l^2 - 1} \prod_{l \nmid r} \frac{l(l^2 - l - 1)}{(l - 1)(l^2 - 1)}.$$

The O -constant depends on c and r .

Theorem 1 cannot be deduced from this since the result of Theorem 6 is not uniform in r . In fact, (6.1) does not hold for large r . It is quite easy to show that, for $t \geq 1$, one has

$$\sum_{\substack{r \leq t \\ 2|r}} C_r = \frac{1}{\pi} t + O(1)$$

Let \mathcal{S} be the set of all odd integers. If assuming that (6.1) were uniform, then for $0 < \alpha < 1$, by partial summation, we have

$$\text{av.}\pi_E(\alpha, \mathcal{S}; X) \sim \sum_{\substack{r \leq 2\alpha\sqrt{X} \\ 2 \nmid r}} C_r(\pi_{1/2}(X) - \pi_{1/2}(r^2/(2\alpha)^2)) \sim \frac{1}{\pi} \alpha \pi(X).$$

This can not be true because, following our argument, it is easy to derive that

$$\text{av.}\pi_E(\alpha, \mathcal{S}; X) \sim \left(\frac{2}{3\pi} \int_0^\alpha \sqrt{1-t^2} dt \right) \pi(X).$$

Another remark we want to make is that in Theorem 1, also the theorems in [1], [5], the minimal ranges of A and B can be reduced a little by estimating some exponential sums. This is of independent interest but not our focus in this paper. So we have not done so.

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