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COMPUTING THE INTEGER PARTITION FUNCTION

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Abstract. In this paper we discuss efficient algorithms for computing the values of the partition function and implement these algorithms in order to conduct a numerical study of some conjectures related to the partition function. We present the distribution of $p(N)$ for $N \leq 10^9$ for primes up to 103 and small powers of 2 and 3.

1. Introduction

Here we discuss some open questions concerning the partition function and algorithms for efficiently computing $p(n)$ for all n up to some bound N. We then present some computational evidence related to these conjectures.

A partition of a natural number n is a non-increasing sequence of natural numbers whose sum is n. The number of such partitions of n is denoted $p(n)$. For example the partitions of 4 are:

$$\begin{aligned} &4, \\ &3+1, \\ &2+2, \\ &2+1+1 \qquad \text{and} \\ &1+1+1+1. \end{aligned}$$

Thus, $p(4) = 5$.

One way of studying the partition function is to study its generating function. Euler [8] proved the following formula concerning this generating function.

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$$(1)\quad P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots .$$

Euler's pentagonal number theorem asserts further that

$$\prod_{n\geq 1}(1-q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(3n^2+n)/2}, \tag{2}$$

from which we immediately deduce the recurrence which we will refer to as Euler's algorithm for computing $p(n)$,

$$(3)\quad p(n) = \sum_{k\geq 1}(-1)^{k+1}\left(p\left(n - \frac{k(3k+1)}{2}\right) + p\left(n - \frac{k(3k-1)}{2}\right)\right).$$

In fact, a careful analysis of the generating function for $p(n)$ leads one to the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \tag{4}$$

which was improved by Rademacher [22] to an exact formula for $p(n)$ (see chapter 5 of [8] for a nice treatment of the expansion of $p(n)$).

One would hope that the presence of an exact formula for $p(n)$ would lead to a good understanding of the partition function or at least to efficient algorithms for the computation of $p(n)$. Indeed, if one desires the value of $p(n)$ for a single value of n then the exact formula of Rademacher yields a very fast algorithm. However, if one wishes to compute $p(n)$ for all $n \leq N$ then Euler's algorithm is much faster. This is because once one already knows $p(1), p(2), \ldots, p(n-1)$, the Euler algorithm only requires $\sqrt{n}$ additions to compute $p(n)$ while the Rademacher formula requires that one compute the sum of $\sqrt{n}$ values of some quite complicated functions. Indeed, many questions concerning the partition function remain open and it is still computationally difficult to compute the values of $p(n)$ for all values of n less than some bound N when N is large.

We now outline some open questions and conjectures concerning the partition function for which we would like to gather numerical evidence. One of the simplest questions that one could ask is the frequency with which $p(n)$ takes on even or odd values. Parkin and Shanks [21] studied this question and were lead to the following conjecture.

Conjecture 1.1. *As $n \to \infty$, we have*

$$\lim_{X\to\infty} \frac{\#\{n \leq X : 2|p(n)\}}{X} = \frac{1}{2}.$$

One can of course ask more generally about the distribution of $p(n)$ modulo an arbitrary modulus. In this direction Newman [17] made the following conjecture.

Conjecture 1.2. *If M is a positive integer, then in every residue class r modulo M there are infinitely many integers n for which*

$$p(n) \equiv r \pmod{M}.$$

Clearly the presence of the Ramanujan-type congruences bears on this question. Ramanujan [23] proved that

$$p(5n+4) \equiv 0 \pmod{5}, \tag{5}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{6}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{7}$$

for all $n \in \mathbb{N}$. The congruence

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13} \tag{8}$$

was discovered in the late 1960's by Atkin, O'Brien and Swinnerton-Dyer (see [9], [10] and [11]). Recently, Ono (see [19] and [24]) and Ahlgren and Ono [7] have proved the existence of infinitely many other such congruences modulo any prime $\ell \geq 5$ while Ahlgren and Boylan [3] have proved that there are no other congruences of this type which are as simple as those of Ramanujan.

In light of the above mentioned congruences, one might expect that the distribution of $p(n)$ would be slightly biased to the zero class modulo a given integer M and otherwise uniform. Let us be more precise.

Definition 1.1. *Let $M \in \mathbb{Z}$ and $0 \leq r \leq M-1$ and define for any $X \in \mathbb{R}$*

$$\delta_r(M, X) := \frac{\#\{0 \leq n < X : p(n) \equiv r \pmod{M}\}}{X}.$$

Then we have the following conjecture of Ahlgren and Ono [6]

Conjecture 1.3. *Let $M \in \mathbb{Z}$, $0 \leq r \leq M-1$ and let $\delta_r(M,X)$ be defined as above. Then,*

(1) *If* $0 \leq r < M$*, then there is a real number* $0 < d_r(M) < 1$ *such that*
$$\lim_{X\to\infty} \delta_r(M,X) = d_r(M).$$
(2) *If* $s \geq 1$ *and* $M = 2^s$*, then for every* $0 \leq i < 2^s$ *we have*
$$d_i(2^s) = \frac{1}{2^s}.$$
(3) *If* $s \geq 1$ *and* $M = 3^s$*, then for every* $0 \leq i < 3^s$ *we have*
$$d_i(3^s) = \frac{1}{3^s}.$$
(4) *If there is a prime* $\ell \geq 5$ *for which* $\ell | M$*, then for every* $0 \leq r < M$ *we have*
$$d_r(M) \neq \frac{1}{M}$$

In the direction of Conjecture 1.1, the best known result is due to Serre [18] and to Ahlgren [1]. We know that
$$\#\{n \leq X : 2|p(n)\} \gg \sqrt{X} \qquad \text{and}$$
$$\#\{n \leq X : 2 \not| p(n)\} \gg \frac{\sqrt{X}}{\log X},$$
which is far from an affirmation of Conjecture 1.1.

In the direction of Newman's Conjecture 1.2, Atkin [9], Kolberg [15], Newman[17] and Klove [16] proved the conjecture for $M = 2$, 5, 7, 13, 17, 19, 29 and 31. Some conditional results were obtained in work of Ono [19], Ahlgren [2], and Bruinier and Ono [12]. Recently, Ahlgren and Boylan [3, 4] have shown that the conjecture is true for $M = \ell^j$ for all primes $\ell \geq 5$ and $j \geq 1$.

Like Conjecture 1.1, Conjecture 1.3 is still wide open. Part (1) of this conjecture is not known for any values of r and M. Theorem 2 of [6] implies that if M is coprime to 6 then
$$\liminf_{X\to\infty} \delta_0(M,X) > 0.$$
This is not known for any other (r, M) pairs. Thus it is would be of interest to gather numerical evidence on all of these conjectures. Additionally, the authors of [6] have shown that the congruences of $p(n)$ are far more widespread than previously known. In particular, [6] has shown that for every prime $l \geq 5$ and any positive integer m, there exist infinitely many arithmetic progressions of the form
$$p(An + B) \equiv 0 \pmod{l^m}$$

for every $n \in \mathbb{Z}$. To be more precise, for each prime $l \geq 5$, define the two integers ϵ_l and δ_l to be

$$\epsilon_l = \left(\frac{-6}{l}\right)$$

and

$$\delta_l = \frac{l^2 - 1}{24}.$$

Further, let S_l be the set of $(l+1)/2$ integers

$$S_l = \left\{\beta \in \{0, 1, \ldots, l-1\} : \left(\frac{\beta + \delta_l}{l}\right) = 0 \text{ or } -\epsilon_l\right\}.$$

Then, we have the following theorem from [7].

Theorem 1.1. *If $l \geq 5$ is prime, m is a positive integer, and $\beta \in S_l$, then there are infinitely many non-nested arithmetic progressions $\{An + B\} \subset \{ln + \beta\}$ such that*

$$p(An + B) \equiv 0 \pmod{l^m}$$

for every integer n.

This naturally leads the authors of [7] to the following speculation [6].

Speculation 1.1. *If $l \geq 5$ is prime and $0 \leq r < l$, define $\delta_r'(l, X)$ by*

$$\delta_r'(l, X) = \frac{\#\{n < X : p(n) \equiv r \pmod{l} \text{ and } n \pmod{l} \notin S_l\}}{\#\{n < X : n \pmod{l} \notin S_l\}}$$

is it true that $\lim_{X \to \infty} \delta_r'(l, X) = \frac{1}{l}$?

It is of additional interest to investigate this speculation numerically.

In the remainder of this paper we discuss a new algorithm for computing the values of the partition function and discuss its running time. We compare the running time of this new algorithm with that of Euler's algorithm. We then discuss a parallelization of the Euler algorithm for computing $p(n)$ modulo a small prime p. We also discuss the scalability of the Euler algorithm. Finally, we present data related to the various conjectures mentioned above.

2. FFT Inversion of Power Series.

To the authors' knowledge the algorithm discussed in this section is new: it is exactly analogous to Euler's proof that the generating functions for partitions into odd parts and for partitions into even parts are identical.

For any function $f(z)$, the function $f(z)f(-z)$ is even. Hence, if $f(z) = \sum a_n z^n$ is a power series and if we write $f(z)f(-z) = \sum b_n z^n$, then b_n is nonzero only if n is even. This leads us to the following expansion for $\frac{1}{f(z)}$.

$$\frac{1}{f(z)} = \frac{f(-z)}{f(z)f(-z)} = \frac{f(-z)}{f_1(z^2)}, \tag{9}$$

where $f_1(z) = \sum b_{2n} z^n$. Now, again $f_1(z)f_1(-z)$ has only even coefficients and thus in the power series expansion of $f_1(z^2)f_1(-z^2)$ the only nonzero coefficients are the coefficients of z^n where $4|n$. Thus, we have

$$\frac{1}{f(z)} = \frac{f(-z)}{f_1(z^2)} = \frac{f(-z)f_1(-z^2)}{f_1(z^2)f_1(-z^2)} = \frac{f(-z)f_1(-z^2)}{f_2(z^4)}. \tag{10}$$

Repeating the above process k times yields

$$\frac{1}{f(z)} = \frac{f_0(-z)f_1(-z^2)\dots f_k(-z^{2^k})}{f_{k+1}(z^{2^{k+1}})}, \tag{11}$$

where

$$\begin{aligned} f_0(z) &= f(z) \qquad \text{and} \\ f_i(z^2) &= f_{i-1}(z)f_{i-1}(-z) \qquad i > 0. \end{aligned} \tag{12}$$

Thus we have that

$$\frac{1}{f(z)} = f_0(-z)f_1(-z^2)\dots f_k(-z^{2^k}) + \mathrm{O}(z^{2^{k+1}}). \tag{13}$$

Thus if one would like to compute the first N coefficients of $\frac{1}{f(z)}$, it suffices to compute the first N coefficients of the product

$$f_0(-z)f_1(-z^2)\dots f_k(-z^{2^k})$$

where $k = \lceil \lg N - 1 \rceil$. One should also note that since we are interested only in the first N coefficients of $\frac{1}{f(z)}$, only the first $\lfloor N/2^i \rfloor$ coefficients of f_i are needed for this computation. In fact, we can say more precisely for any $0 \le \ell \le k$ that only the first $\lfloor N/2^{k-\ell} \rfloor$ coefficients of each of the partial products

$$f_k(-z^{2^k})f_{k-1}(-z^{2^{k-1}}) \cdots f_{k-\ell}(-z^{2^{k-\ell}})$$

are needed in this computation.

Definition 2.1. *Let R be any ring. We define the truncation operator $T_k : R[[z]] \to R[z]$ by*

$$T_k(\sum_{i=0}^{\infty} a_i z^i) = \sum_{i=0}^{k-1} a_i z^i \tag{14}$$

Put $\bar{f}_i = T_{\lfloor N/2^i \rfloor}(f_i)$ and define

$$
(15) \qquad \begin{aligned} g_0(z) &= \bar{f}_k(z) \qquad \text{and} \\ g_i(z) &= T_{\frac{N}{2^{k-i}}}\left(g_{i-1}(-z^2)\bar{f}_{k-i}(z)\right) \qquad 1 \le i \le k. \end{aligned}
$$

Then

$$
(16) \quad g_k(-z) = \bar{f}_0(-z)\bar{f}_1(-z^2)\dots\bar{f}_k(-z^{2^k}) + \mathrm{O}(z^N) = \frac{1}{f(z)} + \mathrm{O}(z^N).
$$

Theorem 2.1. *Let $f(z)$ be a power series with integral coefficients and constant coefficient 1. Let p be any prime which is congruent to 1 modulo $2^{\lceil \lg N \rceil + 1}$. The computation of the first N coefficients of $1/f(z)$ modulo p requires at most $\mathrm{O}(N \lg N)$ coefficient multiplications.*

Proof. It suffices to show that the computation of $g_k(-z)$ described above requires only $\mathrm{O}(N \lg N)$ coefficient multiplications where all arithmetic is done in $\mathbb{F}_p$ and the $\bar{f}_i$'s and g_k's are understood to be in $\mathbb{F}_p[z]$. There are two steps in this computation. First, one must compute $\bar{f}_1(z), \bar{f}_2(z), \dots, \bar{f}_k(z)$ where $k = \lg N - 1$ using (12). Next one must iteratively construct the g_i's (i=0,1,...,k) using (15). We recall that $\deg(\bar{f}_i(z)) = \lfloor N/2^i \rfloor - 1$ and that using the discrete fast Fourier transform, the computation of the product of two polynomials of degree less than M modulo p where $M < N$ requires $\mathrm{O}(M \lg M)$ coefficient multiplications. Thus the the number of coefficient multiplications required for the computation of the $\bar{f}_i$'s is

$$
(17) \qquad \mathrm{O}\left(\sum_{i=0}^{k-1} \frac{N}{2^i} \lg(\frac{N}{2^i})\right) = \mathrm{O}\left(N \lg N\right).
$$

Now we consider the computation of the g_i's. From (15), we have that $\deg(g_{i-1}(-z^2)) < \frac{N}{2^{k-i}}$ and from (12), $\deg(\bar{f}_{k-i})(z) = \frac{N}{2^{k-i}}$. Thus the number of coefficient multiplications required to compute the g_i's is at most

$$
(18) \qquad \mathrm{O}\left(\sum_{i=0}^{k-1} \frac{N}{2^{k-i}} \lg(\frac{N}{2^{k-i}})\right) = \mathrm{O}\left(N \lg N\right)
$$

and this completes the proof of the proposition.

3. Computing $p(n)$.

Theorem 3.1. *Let p be any prime which is congruent to 1 modulo $2^{\lceil \lg N \rceil + 1}$. The computation of $p(n)$ modulo p for all $1 \le n \le N$ requires at most $\mathrm{O}(N \lg N \lg p \lg \lg p)$ machine multiplications.*

Proof. Combining (1) and (2), we see that

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\sum_{n \in \mathbb{Z}} (-1)^n q^{(3n^2+n)/2}} \tag{19}$$

The computation of the first N terms of the denominator clearly requires at most $\mathrm{O}(\sqrt{N})$ multiplications of integers which are easily bounded in absolute value by $\sqrt{N}$. Using the discrete FFT, one can perform multiplication of integers of size $\sqrt{N}$ using at most $\mathrm{O}(\sqrt{N} \lg N)$ machine multiplications. Thus, the first N terms of the denominator can be computed using at most $\mathrm{O}(N \lg N)$ machine multiplications. Now we may use our inversion algorithm which will require an additional $\mathrm{O}(N \lg N)$ coefficient multiplications. Since, we are working modulo p, each coefficient may be taken to be between 0 and $p-1$. Thus using discrete FFT, the multiple of any two coefficients requires at most $\mathrm{O}(\lg p \lg \lg p)$ machine multiplications. Thus the first N coefficients of $P(q)$ modulo p requires at most $\mathrm{O}(N \lg N + N \lg N \lg p \lg \lg p) = \mathrm{O}(N \lg N \lg p \lg \lg p)$ machine multiplications.

Corollary 3.1. *The computation of $p(n)$ for all $n \le N$ can be done using $\mathrm{O}(N^{3/2} \log^2(N))$ machine multiplications.*

Proof. Select a prime p satisfying $p(N) < p < (4Np(N))^{5.5}$ and $p \equiv 1 \pmod{2^{\lceil \lg N \rceil + 1}}$. To see that this can be done we note that the main result of [14] guarantees that for $(A, B) = 1$, there is a prime congruent to A modulo B that is smaller than $B^{5.5}$. Taking $A = 1$ and $B = 2^{\lceil \lg N \rceil + 1} p(N)$ then guarantees us that there is a prime p as above since $2^{\lceil \lg N \rceil + 1} < 4N$. By Theorem 3.1 one can compute $p(n)$ modulo p for all $1 \le n \le N$ using $\mathrm{O}(N \lg N \lg p \lg \lg p)$ machine multiplications. However, since $0 < p(n) < p$ this gives the exact value of $p(n)$ for all $1 \le n \le N$. Using (4), one can see that for N sufficiently large, we may assume that $p < e^{15\sqrt{N}}$. Combining this with our previous estimate on the number of machine multiplications yields the desired result.

We will refer to the algorithm suggested by the previous two results as fast Fourier Transform inversion or simply as FFTI. A careful analysis of the Euler algorithm yields the following results.

Theorem 3.2. *The computation of $p(n)$ for all $1 \le n \le N$ using the Euler algorithm requires $\mathrm{O}(N^2)$ machine additions.*

Proof. Examining (3), one sees that for each $1 \le n \le N$ the computation of $p(n)$ requires the addition of $n^{1/2}$ values of the partition function. Thus the computation of $p(n)$ for all $1 \le n \le N$ requires $n^{3/2}$ additions of numbers which grow as large as $p(N)$. The theorem now follows from (4) $\square$

Theorem 3.3. *For $m \in \mathbb{N}$, the computation of $p(n)$ modulo m for all $1 \le n \le N$ requires at most $\mathrm{O}(N^{3/2} \log m)$ machine additions.*

Proof. The proof is the same as that of the previous theorem. $\square$

The FFTI algorithm has a faster running time than the Euler algorithm for computing $p(n)$ for all $1 \le n \le N$. However, if one is interested only in the values of $p(n)$ modulo m where m is not a prime which is 1 modulo $2^{\lceil \lg N \rceil + 1}$ and where $\log(m) < \log^2(N)$ then the Euler algorithm has the faster running time. We now consider a parallel version of the Euler algorithm that improves the running time by a constant factor when the later case is of interest.

4. A parallel version of Euler's algorithm

We now consider a parallel implementation of Euler's algorithm for computing $p(n)$ for all $1 \le n \le N$. When computing in parallel one wishes to equally distribute the computation and data storage among all, say N_p, processors. Since each partition number requires summing a subset of the previously computed numbers one can not store partition numbers linearly across the processors and expect balanced work across all N_p processors at any given instance. Thus we store the partition number $p(n)$ on the i_p^{th} processor if and only if $n \equiv i_p \pmod{N_p}$. Now, we make the following definitions.

Definition 4.1. *Let $m, n \in \mathbb{N}_0$ and $m \le n$, then the partial Euler sum of n up to m, $\tilde{p}_m(n)$, is*

$$\tilde{p}_m(n) = \sum_{\substack{k \\ 0 \le n - k(3k+1)/2 \le m}} (-1)^{k+1} p\left(n - \frac{k(3k+1)}{2}\right) + \sum_{\substack{j \\ 0 \le n - j(3j-1)/2 \le m}} (-1)^{j+1} p\left(n - \frac{j(3j-1)}{2}\right).$$

Definition 4.2. *Let $i, m, n, N \in \mathbb{N}_0$, $i < N$ and $m \leq n$, then the i^{th} N-wise decimation of the partial Euler sum of n up to m, $\tilde{q}_{m,i,N}(n)$, is*

$$\tilde{q}_{m,i,N}(n) = \sum_{\substack{k \\ n-k(3k+1)/2 \equiv i \pmod N \\ 0 \leq n-k(3k+1)/2 \leq m}} (-1)^{k+1} p\left(n - \frac{k(3k+1)}{2}\right) + \sum_{\substack{j \\ n-j(3j-1)/2 \equiv i \pmod N \\ 0 \leq n-j(3j-1)/2 \leq m}} (-1)^{j+1} p\left(n - \frac{j(3j-1)}{2}\right).$$

Note that $\sum_{i=0}^{N-1} \tilde{q}_{m,i,N}(n) = \tilde{p}_m(n)$ and $\tilde{p}_{n-1}(n) = p(n)$. Thus, to compute $p(n)$ distributed across N_p processors we can compute $\tilde{q}_{n-1,i_p,N_p}(n)$ on the i_p^{th} processor in parallel and upon completion, $\tilde{p}_{n-1}(n)$ on one processor. We now discuss how to efficiently compute the sums appearing in Definition 4.2.

In order to efficiently compute $\tilde{q}_{m,i,N}(n)$ we note that one does not need to check the condition $n - k(3k \pm 1)/2 \equiv i \pmod N$ for all values of k. By completing the square, we have $n - k(3k \pm 1)/2 \equiv i \pmod N$ if and only if

$$6k \equiv \mp 1 + \sqrt{24(n-i)+1} \pmod N \tag{20}$$

or

$$6k \equiv \mp 1 - \sqrt{24(n-i)+1} \pmod N, \tag{21}$$

provided that this square root exists. In this direction we provide the following definitions.

Definition 4.3. *Let $i, N \in \mathbb{Z}$, N prime, $N \neq 2, 3$. We define for any $n \in \mathbb{Z}$,*

$$\chi_{i,N}(n) = \left(\frac{24(n-i)+1}{N}\right)$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol.

Definition 4.4. *Let $n, i, N \in \mathbb{N}_0$, N prime, $N \neq 2, 3$, and $\chi_{i,N}(n) \neq -1$. Then, we define*

$$t_{i,N}(n) = \sqrt{24(n-i)+1} \pmod N \tag{22}$$

where the square root in (22) is the unique integer r, $0 \leq r < (N-1)/2$, such that $r^2 = 24(n-i)+1 \pmod N$.

Thus, if $n - k(3k \pm 1)/2 \equiv i \pmod N$ then

(23) $6k \equiv \mp 1 + t_{i,N}(n) \pmod N$ or $6k \equiv \mp 1 - t_{i,N}(n) \pmod N$

Combining (23),(20) and (21) leads to the following definitions.

Definition 4.5. *Let $n, i, N \in \mathbb{N}_0$, N prime, $N \neq 2,3$, and $\chi_{i,N}(n) \neq -1$. We define for any $l \in \mathbb{Z}$,*

$$\begin{aligned} \alpha_{0,i,N}(l,n) &= -6^{-1}\left(1 - t_{i,N}(n)\right) + lN \\ \alpha_{1,i,N}(l,n) &= 6^{-1}\left(1 + t_{i,N}(n)\right) + lN \\ \alpha_{2,i,N}(l,n) &= -6^{-1}\left(1 + t_{i,N}(n)\right) + lN \\ \alpha_{3,i,N}(l,n) &= 6^{-1}\left(1 - t_{i,N}(n)\right) + lN \end{aligned}$$

and for $j \in \{0,1,2,3\}$

$$\rho_{j,i,N}(l,n) = n - \frac{\alpha_{j,i,N}(l,n)(3\alpha_{j,i,N}(l,n) + (-1)^j)}{2}$$

where $0 \leq 6^{-1} < N$ is the unique integer such that $6 \cdot 6^{-1} \equiv 1 \pmod N$.

We will omit the subscripts i, N in Definitions 4.3 and 4.5 when they are apparent in the context they are used. Definition 4.2 may now be rewritten as

(24)
$$\tilde{q}_{m,i,N}(n) = \begin{cases} \displaystyle\sum_{j=0}^{3} \sum_{\substack{l \\ 0 \leq \rho_j(l,n) \leq m}} (-1)^{\alpha_j(l,n)+1} p\left(\rho_j(l,n)\right) & if\ \chi(n) = 1 \\ \displaystyle\sum_{j=0}^{1} \sum_{\substack{l \\ 0 \leq \rho_j(l,n) \leq m}} (-1)^{\alpha_j(l,n)+1} p\left(\rho_j(l,n)\right) & if\ \chi(n) = 0 \\ 0 & if\ \chi(n) = -1 \end{cases}$$

Note that $\tilde{q}_{m,i,N}(n)$ vanishes whenever $\chi(n) = -1$. Thus, since N_p is a prime not 2 or 3, $\chi_{i_p,N_p}(n) \neq -1$ for exactly $(N_p+1)/2$ many $0 \leq i_p < N_p$ for any given $n \geq N_p$. Thus, for any computation of $p(n)$ only $(N_p+1)/2$ processors are used. However, since N_p is coprime to 6, $2(n-i)3^{-1} + 36^{-1}$ runs through all equivalence classes mod N_p for N_p consecutive n. Thus, if $\tilde{q}_{m,i,N}(n)$ is computed for L consecutive values of n at any instance, where $L = jN_p$ and $j \in \mathbb{N}_0$, each processor computes exactly $j(N_p+1)/2$ non-zero $\tilde{q}_{m,i,N}(n)$. We now show how this can be used to compute $p(n)$.

Begin by letting n_k be largest value for which $p(n)$ is know exactly. On any processor, say i, we may then compute $\tilde{q}_{n_k,i,N}(n_k+1), \tilde{q}_{n_k,i,N}(n_k+2), \ldots, \tilde{q}_{n_k,i,N}(n_k+L)$. Upon completion, one processor may then compute $\tilde{p}_{n_k}(n_k+1), \tilde{p}_{n_k}(n_k+2), \ldots, \tilde{p}_{n_k}(n_k+L)$. Then, since $\tilde{p}_{n_k}(n_k+1) = p(n_k+1)$ we may use this processor to compute $p(n_k+1), p(n_k+2), \ldots, p(n_k+L)$. We call this one processor the control processor. This could be a separate processor in addition to the N_p processors already in use. Our algorithm to compute $p(n)$ is the following:

Algorithm 4.1. *For any integer $j > 0$ and $N_p > 0$, set $L = jN_p$. On every processor except the control processor:*

(0) *Set $l = 0$*
(1) *Have each processor compute $p(n)$ exactly for $0 \le n < L$*
(2) *Set $n_k = L - 1$, $l = 1$*
(3) *Compute $\tilde{q}_{n_k,i_p,N_p}(n)$ on each processor, i_p, using (24) for $(l+1)L \le n < (l+2)L$.*
(4) *If $l \neq 1$, receive exact values for $p(n)$ for $lL \le n < (l+1)L$ and set $n_k = n_k + L$*
(5) *Finish Computing $\tilde{q}_{n_k,i_p,N_p}(n)$ on each processor i_p using (24) for $(l+1)L \le n < (l+2)L$.*
(6) *Send all computed $\tilde{q}_{n_k,i_p,N_p}$ to control process*
(7) *Set $l = l + 1$ goto 3.*

On the control process we follow:

(0) *Set $l = 1$, $n_k = L - 1$*
(1) *Receive $\tilde{q}_{n_k,i_p,N_p}(n)$ from all processors for $lL \le n < (l+1)L$.*
(2) *Compute $\tilde{p}_{n_k}(n) = \sum_{i=0}^{N_p-1} \tilde{q}_{n_k,i_p,N_p}(n)$ for $lL \le n < (l+1)L$*
(3) *Set $p(n_k+1) = \tilde{p}_{n_k}(n_k+1)$.*
(4) *Compute $p(n)$ exactly using $\tilde{p}_{n_k}(n)$ and $p(j)$ for $n_k \le j < n$ for $lL \le n < (l+1)L$*
(5) *Send exact $p(n)$ to all processors*
(6) *Set $l = l + 1$, $n_k = n_k + L$, goto 1*

5. Discussion

The algorithm of Section 4 was used with $N_p = 108$ to compute $p(n)$ modulo primes less than 104 for $n \le 10^9$. We list the statistical properties of the partition function up to 10^9 in the Appendix. Computations are ongoing and further data can be found at [25]. In order to be concise we only list the intermediate results for $10^6, 10^7, 10^8$ and 10^9, and plot the cases in which we wish to be more precise.

In regard to Conjecture 1.1 we see that the conjecture is justified. Examining Table 1 we see that the distribution agrees out to the 4th decimal place. Similarly for $M = 3$, Table 2 shows that the distribution is $1/3$ out to 4 decimal places.

	10^6	10^7	10^8	10^9
$\delta_0(2,X)$	0.500447	0.499786	0.500029	0.500036
$\delta_1(2,X)$	0.499554	0.500214	0.499971	0.499964

Table 1: The Distribution of $p(n) \pmod 2$

	10^6	10^7	10^8	10^9
$\delta_0(3,X)$	0.333013	0.333163	0.333287	0.333325
$\delta_1(3,X)$	0.333629	0.333630	0.333414	0.333335
$\delta_2(3,X)$	0.333359	0.333207	0.333299	0.333340

Table 2: The Distribution of $p(n) \pmod 3$

In order to examine the conjectures and speculation of [6] we make the following definitions.

Definition 5.1. *Let $M \in \mathbb{Z}$ and define for any $X \in \mathbb{Z}$*

$$\mu_d(M,X) = \frac{1}{M-1}\sum_{j=1}^{M-1}\delta_j(M,X) = \frac{1-\delta_0(M,X)}{M-1}$$

and

$$\sigma_d^2(M,X) = \frac{1}{M-1}\sum_{j=1}^{M-1}\left(\delta_j(M,X)-\mu_d(M,X)\right)^2$$

to be the mean and variance of the distribution of $p(n) \pmod M$ among the non-zero congruence classes mod M for $n < X$.

In regard to Conjecture 1.3 (1) the computational evidence suggests that this conjecture is justified. Examining Table 3 in the Appendix we can see that for primes $M \leq 103$ it appears that the distribution of $p(n) \pmod M$ in the zero class is approaching a limit, say $d_0(M)$. Additionally, by examining Table 4 in the Appendix we can see that the variance of the distribution $\delta_j(M,X)$ is tending to zero, implying that the distribution of $p(n) \pmod M$ approaches a limit. In fact, since the variance is tending to zero the computations suggest that $p(n)$ is equally likely to lie in any of the non-zero classes modulo M. Thus, if (as the computations suggest) $\lim_{X\to\infty}\delta_0(M,X) = d_0(M)$ and $\lim_{X\to\infty}\sigma_d(M,X) = 0$ then

$$\lim_{X\to\infty}\delta_j(M,X) = \frac{1-d_0(M)}{M-1} \quad \forall \quad 0<j<M. \tag{25}$$

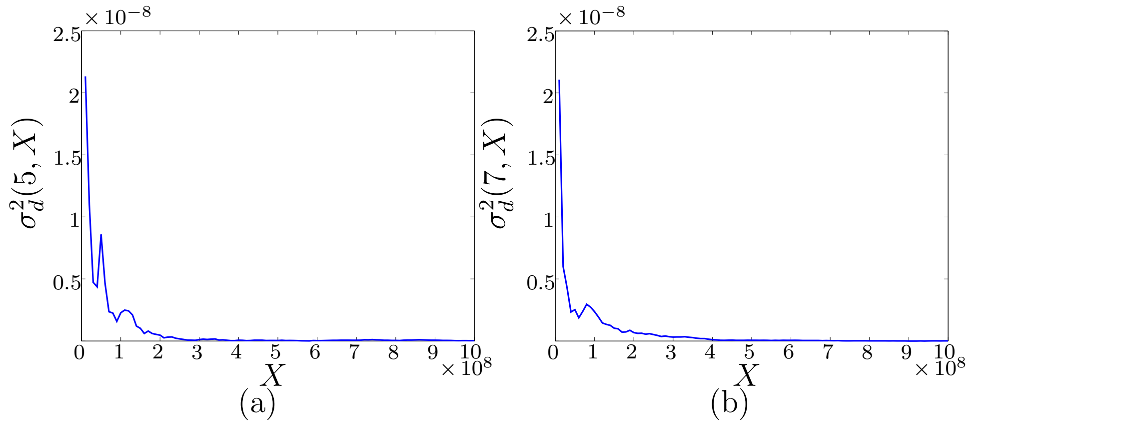


FIGURE 1. The variance of δ_j for j not 0. For (a) $M = 5$ and (b) $M = 7$ and X from 1000 to 10^9 in steps of 1000.

Table 4 also suggests that not only does the distribution of $p(n) \pmod M$ converge, but it converges at a very fast rate. Indeed, examining Figure 1 for $M = 5$ and $M = 7$ it appears that the variance is tending to 0 at an exponential rate. Further, examining Table 4 for all other primes this same trend appears. That is, as X grows linearly so does the exponent of the variance. This leads us to the following speculation.

Speculation 5.1. *Is it true that for any $M \geq 3$ and for any $0 \leq j \leq M - 1$*

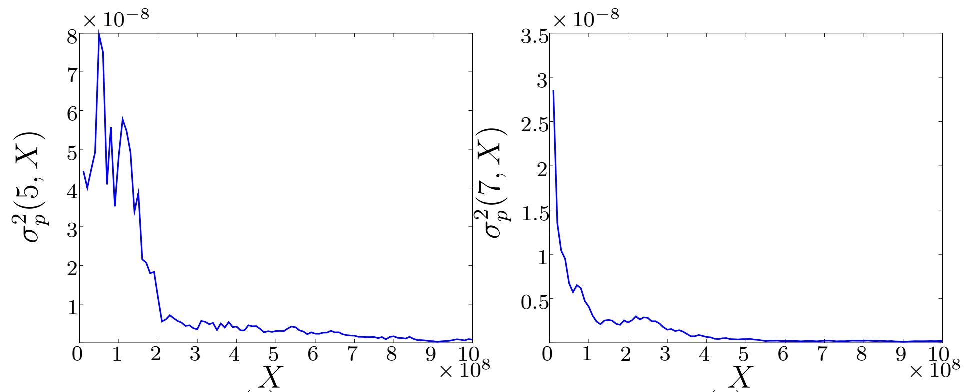
$$\lim_{X \to \infty} \frac{-\log |\delta_j(M, X) - d_j(M)|}{X} > 0 \text{ ?}$$

As previously noted, due the congruence properties of $p(n)$, it is reasonable to think that $p(n)$ is biased toward the zero class. That is, $d_0(M) > d_j(M)$ for $0 < j \leq M - 1$. It is reasonable to expect that this holds for all prime powers $M > 5$ since, by Theorem 1.1, there are infinitely many non-nested arithmetic progressions

$$p(An + B) \equiv 0 \pmod M.$$

However, if $d_0(M) > 1/M$ and (25) is true, this would imply that $d_j(M) < 1/M$ for $0 < j \leq M - 1$. Examining Table 3 for small prime $M > 5$ it appears that indeed $d_0(M) > 1/M$ and thus Conjecture 1.3 (4) is also justified. However, since the known congruences modulo M for $M > 11$ have $A \gg 1$ the influence of these progressions are not apparent in the computation for all primes.

It is natural to consider how one may remove the bias of $p(n)$ to the zero class. That is, it is natural to consider if one can, by excluding a subset of integers, say S, make the distribution of $p(n) \pmod M$



(a) (b)

FIGURE 2. The variance of the the distribution of the congruence classes for $p(n)$ where $n \notin S_M$. For (a) $M = 5$ and (b) $M = 7$ and X from 1000 to 10^9 in steps of 1000.

uniform for $n \notin S$. This is the content of Speculation 1.1. Recall that for prime $M \geq 5$, $\delta'_r(M, X)$ is defined to be the distribution of $p(n) \pmod M$ for $n \pmod M \notin S_M$. That is,

$$\delta'_r(M, X) = \frac{\#\{n < X : p(n) \equiv r \pmod M \text{ and } n \pmod M \notin S_M\}}{\#\{n < X : n \pmod M \notin S_M\}}.$$

Then, similar to Definition 5.1 we have the following definitions concerning the distribution of $p(n) \pmod M$ for $n \notin S_M$.

Definition 5.2. *Let M be a prime with $M \geq 5$ and define for any $X \in \mathbb{Z}$*

$$\mu_p(M, X) = \frac{1}{M} \sum_{j=0}^{M-1} \delta'_j(M, X) = \frac{1}{M}$$

and

$$\sigma_p^2(M, X) = \frac{1}{M} \sum_{j=0}^{M-1} \left(\delta'_j(M, X) - \mu_p(M, X)\right)^2$$

to be the mean and variance of the distribution of $p(n) \pmod M$ for $n \notin S_M$ and $n < X$ respectively.

The computed values for $\sigma_p^2(M, X)$ can be seen in Table 5 in the Appendix. The speculation of [6] seems to well justified. In fact the variance of the distribution of $p(n)$ for $n \notin S_M \pmod M$ again appears to decay exponentially in X. This can be seen in Figure 2 (a) and (b) for $M = 5$ and $M = 7$. By examining Table 5 we can see that this trend appears for all computed primes. This leads us to the following speculation.

Speculation 5.2. *Is it true that for any prime $M \geq 5$ and for any $0 \leq j \leq M-1$*

$$\lim_{X\to\infty} \frac{-\log|\delta'_j(M,X) - 1/M|}{X} > 0 \text{ ?}$$

A notable exception to the congruence properties of $p(n)$ appear to occur for a modulus which is a power of 2 or 3. That is, if Conjecture 1.3 (2) and (3) are true then $p(n)$ is not biased toward the zero class modulo 2^m or 3^m for any $m \in \mathbb{N}$. In this direction we make the following definitions.

Definition 5.3. *Let $M \in \mathbb{Z}$ define for any $X \in \mathbb{Z}$*

$$\mu(M,X) = \frac{1}{M}\sum_{j=0}^{M-1} \delta_j(M,X) = \frac{1}{M}$$

and

$$\sigma^2(M,X) = \frac{1}{M}\sum_{j=0}^{M-1}\left(\delta_j(M,X) - \frac{1}{M}\right)^2$$

to be the mean and variance of the distribution of $p(n) \pmod M$ for $n < X$ respectively.

The computed values for $\delta_0(M,X)$ and $\sigma^2(M,X)$ can be seen in Table 6 and Table 7 in the Appendix for powers of 2 and the computed values for $\delta_0(3^m,X)$ and $\sigma^2(3^m,X)$ can be seen in Table 8 and Table 9 in the Appendix. Examining Table 6 and Table 8 we see that Conjecture 1.3 (2) and (3) is well justified. Indeed, for $X = 10^9$ there appears to be no bias toward the zero class as δ_0 agrees to the 5th decimal place. Further, in these cases the variance again appears to tend to zero exponentially fast further supporting Speculation 5.1.

In conclusion, we see that the computations suggest that Conjecture 1.3 and Speculation 1.1 seem to be well justified. In fact, our computations also suggested a stronger result than Conjecture 1.3. This leads us to the following revision of Conjecture 1.3.

Conjecture 5.1. *Let $M \in \mathbb{Z}$, $0 \leq r \leq M-1$. Then, there exists a real number $0 < d(M) < 1$ such that*

$$\lim_{X\to\infty} \delta_0(M,X) = d(M)$$

and $\forall\, 0 < r < M$

$$\lim_{X\to\infty} \delta_r(M,X) = \frac{1-d(M)}{M-1}$$

In particular,

(1) *If* $s \geq 1$ *and* $M = 2^s$, *then* $d(2^s) = 1/2^s$
(2) *If* $s \geq 1$ *and* $M = 3^s$, *then* $d(3^s) = 1/3^s$
(3) *If there is a prime* $\ell \geq 5$ *for which* $\ell | M$, *then* $d(M) \neq 1/M$

6. Acknowledgments

The authors would like to thank Dan Stanzione of the High Performance Computing Initiative, Fulton School of Engineering at Arizona State University and an anonymous reviewer for helpful comments.

Appendix A. Data

	$\delta_0(M, 10^6)$	$\delta_0(M, 10^7)$	$\delta_0(M, 10^8)$	$\delta_0(M, 10^9)$
2	0.500447	0.499786	0.500029	0.500036
3	0.333013	0.333163	0.333287	0.333325
5	0.363677	0.364091	0.364455	0.364610
7	0.272756	0.272939	0.273082	0.273174
11	0.173382	0.173569	0.173523	0.173563
13	0.080236	0.079782	0.079476	0.079252
17	0.058708	0.058871	0.058940	0.058947
19	0.052607	0.052761	0.052865	0.052863
23	0.043637	0.043661	0.043762	0.043760
29	0.034710	0.034564	0.034559	0.034552
31	0.032461	0.032312	0.032240	0.032263
37	0.026949	0.027055	0.027045	0.027026
41	0.024351	0.024370	0.024391	0.024386
43	0.023182	0.023215	0.023226	0.023256
47	0.021223	0.021280	0.021278	0.021270
53	0.018927	0.018847	0.018852	0.018861
59	0.016908	0.016965	0.016942	0.016950
61	0.016296	0.016367	0.016392	0.016400
67	0.014856	0.014941	0.014947	0.014927
71	0.013791	0.014056	0.014081	0.014085
73	0.013593	0.013716	0.013687	0.013694
79	0.012398	0.012631	0.012674	0.012664
83	0.012070	0.012119	0.012056	0.012050
89	0.011113	0.011246	0.011224	0.011231
97	0.010326	0.010354	0.010323	0.010307
101	0.009887	0.009895	0.009913	0.009901
103	0.009626	0.009717	0.009703	0.009705

Table 3. The values of δ_0 for primes ≤ 103

	$\sigma_d^2(M,10^6)$	$\sigma_d^2(M,10^7)$	$\sigma_d^2(M,10^8)$	$\sigma_d^2(M,10^9)$
3	1.823e-08	4.471e-08	3.351e-09	4.930e-12
5	1.040e-07	2.133e-08	2.275e-09	3.481e-11
7	8.820e-08	2.107e-08	2.361e-09	1.351e-11
11	1.156e-07	3.864e-09	4.948e-10	7.233e-11
13	6.045e-08	8.387e-09	9.000e-10	3.670e-11
17	5.529e-08	1.282e-09	5.044e-10	4.091e-11
19	3.396e-08	3.261e-09	4.398e-10	5.140e-11
23	2.775e-08	2.323e-09	3.043e-10	5.312e-11
29	4.029e-08	3.400e-09	4.279e-10	1.974e-11
31	2.570e-08	1.766e-09	3.190e-10	3.041e-11
37	4.034e-08	2.159e-09	2.208e-10	2.678e-11
41	2.189e-08	1.802e-09	2.113e-10	2.315e-11
43	2.058e-08	2.360e-09	2.707e-10	2.574e-11
47	2.279e-08	1.202e-09	2.562e-10	2.032e-11
53	1.829e-08	1.937e-09	1.508e-10	1.664e-11
59	2.271e-08	1.801e-09	1.494e-10	1.225e-11
61	1.948e-08	1.610e-09	1.902e-10	1.504e-11
67	1.761e-08	1.813e-09	1.192e-10	1.272e-11
71	1.286e-08	1.401e-09	1.452e-10	1.293e-11
73	1.332e-08	1.581e-09	1.198e-10	1.595e-11
79	1.137e-08	1.310e-09	1.136e-10	1.379e-11
83	8.823e-09	1.353e-09	1.440e-10	1.056e-11
89	7.730e-09	9.219e-10	1.039e-10	1.188e-11
97	1.098e-08	9.991e-10	1.193e-10	1.220e-11
101	9.600e-09	9.882e-10	5.857e-11	1.124e-11
103	1.121e-08	1.003e-09	7.340e-11	9.771e-12

TABLE 4. The variance of $\delta_j(M,X)$ about $\mu_d(M,X)$ for $j \neq 0$

	$\sigma_p^2(M,10^6)$	$\sigma_p^2(M,10^7)$	$\sigma_p^2(M,10^8)$	$\sigma_p^2(M,10^9)$
5	1.262e-07	4.441e-09	4.835e-09	8.154e-11
7	2.874e-07	2.858e-08	4.090e-09	2.225e-10
11	2.311e-07	1.065e-08	1.464e-09	2.238e-10
13	7.622e-08	1.729e-08	2.140e-09	1.006e-10
17	1.296e-07	9.688e-09	1.387e-09	1.108e-10
19	9.526e-08	9.943e-09	9.045e-10	1.151e-10
23	8.018e-08	8.045e-09	7.891e-10	1.234e-10
29	1.135e-07	6.671e-09	9.132e-10	6.695e-11
31	4.017e-08	3.118e-09	6.685e-10	6.127e-11
37	5.998e-08	4.262e-09	3.480e-10	6.084e-11
41	3.848e-08	3.918e-09	3.035e-10	5.349e-11
43	4.615e-08	5.388e-09	6.182e-10	5.150e-11
47	3.677e-08	2.836e-09	5.205e-10	4.283e-11
53	3.533e-08	4.275e-09	2.914e-10	4.761e-11
59	3.869e-08	3.522e-09	3.430e-10	2.746e-11
61	4.399e-08	3.439e-09	3.016e-10	3.531e-11
67	2.957e-08	3.386e-09	2.754e-10	1.959e-11
71	3.240e-08	3.089e-09	2.672e-10	2.178e-11
73	2.530e-08	2.925e-09	2.968e-10	2.372e-11
79	1.849e-08	2.564e-09	2.565e-10	2.560e-11
83	2.175e-08	2.916e-09	2.908e-10	2.007e-11
89	2.108e-08	2.014e-09	2.389e-10	2.536e-11
97	2.588e-08	1.932e-09	2.216e-10	2.345e-11
101	2.285e-08	2.210e-09	1.700e-10	1.960e-11
103	2.404e-08	2.247e-09	2.065e-10	1.804e-11

Table 5. The variance of $\delta'_j(M,X)$ about $\mu_p(M,X)$

	$\delta_0(M, 10^7)$	$\delta_0(M, 10^8)$	$\delta_0(M, 10^9)$	$1/M$
2^1	0.499786	0.500029	0.500036	0.500000
2^2	0.249832	0.249982	0.250015	0.250000
2^3	0.124911	0.124961	0.125008	0.125000
2^4	0.062438	0.062473	0.062504	0.062500
2^5	0.031165	0.031236	0.031254	0.031250
2^6	0.015598	0.015613	0.015622	0.015625
2^7	0.007790	0.007805	0.007811	0.007813
2^8	0.003896	0.003906	0.003904	0.003906
2^9	0.001927	0.001955	0.001952	0.001953
2^{10}	0.000970	0.000979	0.000977	0.000977
2^{11}	0.000486	0.000490	0.000489	0.000488
2^{12}	0.000236	0.000246	0.000245	0.000244
2^{13}	0.000119	0.000125	0.000123	0.000122
2^{14}	0.000064	0.000064	0.000062	0.000061
2^{15}	0.000031	0.000031	0.000031	0.000031

TABLE 6. The values of δ_0 for powers of 2

	$\sigma^2(M, 10^7)$	$\sigma^2(M, 10^8)$	$\sigma^2(M, 10^9)$
2^1	4.564635e-08	8.343399e-10	1.269179e-09
2^2	2.304500e-08	8.904586e-10	5.107867e-10
2^3	7.611309e-09	1.567789e-09	1.397324e-10
2^4	3.888221e-09	4.386575e-10	6.467568e-11
2^5	3.074581e-09	1.955245e-10	3.558670e-11
2^6	1.585966e-09	1.155879e-10	1.360000e-11
2^7	8.183707e-10	8.195320e-11	6.882936e-12
2^8	3.832744e-10	4.217360e-11	3.472434e-12
2^9	1.949290e-10	2.079980e-11	1.802906e-12
2^{10}	9.566222e-11	9.907638e-12	8.912183e-13
2^{11}	4.831187e-11	4.909129e-12	4.648978e-13
2^{12}	2.387426e-11	2.435416e-12	2.396083e-13
2^{13}	1.191768e-11	1.224673e-12	1.202085e-13
2^{14}	6.026680e-12	6.093979e-13	6.140721e-14
2^{15}	3.072776e-12	3.027812e-13	3.077366e-14

TABLE 7. The variance of $\delta_j(M, X)$ about $1/M$ for powers of 2.

	$\delta_0(M, 10^7)$	$\delta_0(M, 10^8)$	$\delta_0(M, 10^9)$	$1/M$
3^1	0.333163	0.333287	0.333325	0.333333
3^2	0.111198	0.111112	0.111113	0.111111
3^3	0.036975	0.037021	0.037037	0.037037
3^4	0.012315	0.012343	0.012349	0.012346
3^5	0.004127	0.004118	0.004118	0.004115
3^6	0.001369	0.001372	0.001375	0.001372
3^7	0.000453	0.000457	0.000457	0.000457
3^8	0.000153	0.000153	0.000153	0.000152
3^9	0.000053	0.000051	0.000051	0.000051
3^{10}	0.000018	0.000018	0.000017	0.000017
3^{11}	0.000006	0.000006	0.000006	0.000006
3^{12}	0.000002	0.000002	0.000002	0.000002
3^{13}	0.000001	0.000001	0.000001	0.000001

TABLE 8. The values of δ_0 for powers of 3

	$\sigma^2(M, 10^7)$	$\sigma^2(M, 10^8)$	$\sigma^2(M, 10^9)$
3^1	4.438803e-08	3.307325e-09	3.513351e-11
3^2	1.025873e-08	1.622078e-09	1.148162e-10
3^3	4.370097e-09	5.261098e-10	3.191454e-11
3^4	1.198509e-09	1.103418e-10	1.186007e-11
3^5	3.859236e-10	4.232208e-11	3.834449e-12
3^6	1.350973e-10	1.288184e-11	1.413339e-12
3^7	4.493377e-11	4.441386e-12	4.616726e-13
3^8	1.521634e-11	1.519451e-12	1.513390e-13
3^9	5.078601e-12	5.117933e-13	5.134899e-14
3^{10}	1.689233e-12	1.689466e-13	1.684987e-14
3^{11}	5.633204e-13	5.624952e-14	5.670555e-15
3^{12}	1.877057e-13	1.874117e-14	1.885936e-15
3^{13}	6.258333e-14	6.262380e-15	6.277984e-16

TABLE 9. The variance of $\delta_j(M, X)$ about $1/M$ for powers of 3.

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