Divisibility properties of the 5-regular and 13-regular partition functions

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Abstract

The function $b_k(n)$ is defined as the number of partitions of n that contain no summand divisible by k. In this paper we study the 2-divisibility of $b_5(n)$ and the 2- and 3-divisibility of $b_{13}(n)$. In particular, we give exact criteria for the parity of $b_5(2n)$ and $b_{13}(2n)$.

1 Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. In other words,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_t$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1$. For instance, the partitions of 4 are

$$\begin{gathered} 4, \\ 3+1, \\ 2+2, \\ 2+1+1 \text{ and} \\ 1+1+1+1. \end{gathered}$$

We denote the number of partitions of n by $p(n)$. So, as shown above, $p(4) = 5$. Note that $p(n) = 0$ if n is not a nonnegative integer, and we adopt the convention that $p(0) = 1$. The generating function for the partition function is then given by the infinite product

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots .$$

Let k be a positive integer. We say that a partition is *k-regular* if none of its summands is divisible by k, and denote the number of k-regular partitions of n by $b_k(n)$. For example, $b_3(4) = 4$ because the partition $3 + 1$ has a summand divisible by 3 and therefore is not 3-regular. Adopting the convention that $b_k(0) = 1$, the generating function for the k-regular partition function is then

$$\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{\substack{n=1 \\ k\nmid n}}^{\infty} \frac{1}{(1-q^n)} = \prod_{n=1}^{\infty} \frac{(1-q^{kn})}{(1-q^n)}. \tag{1}$$

Note that $b_2(n)$ equals the number of partitions of n into odd parts, which Euler proved is equal to the number of partitions of n into distinct parts.

The partition function satisfies the famous Ramanujan congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

for every $n \geq 0$. Ono [7] proved that such congruences for $p(n)$ exist modulo every prime ≥ 5, and Ahlgren [1] extended this to include every modulus coprime to 6. Given these facts, for a positive integer m it is natural to wonder for which values of n we have that $p(n)$ divisible by m, or simply how often $p(n)$ is divisible by m. By the results cited above,

$$\liminf_{X\to\infty} \#\{1 \leq n \leq X \mid p(n) \equiv 0 \pmod m\}/X > 0$$

for any m coprime to 6. The $m = 2$ and $m = 3$ cases, meanwhile, have proven elusive.

The state of knowledge for k-regular partition functions is better. For example, Gordon and Ono [4] have shown that if p is prime, $p^v \parallel k$ and $p^v \geq \sqrt{k}$, then for any $j \geq 1$ the arithmetic density of positive integers n such that $b_k(n)$ is divisible by p^j is one. In certain cases one can find even more specific information. As an illustration we consider the parity of $b_2(n)$. Noting that

$$\begin{aligned} \sum_{n=0}^{\infty} b_2(n)q^n = \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1-q^n}\right) &\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n)} \\ &\equiv \prod_{n=1}^{\infty} (1-q^n) \pmod 2, \end{aligned}$$

by Euler's Pentagonal Number Theorem it follows that

$$\sum_{n=0}^{\infty} b_2(n)q^n \equiv \sum_{\ell=-\infty}^{\infty} q^{\ell(3\ell+1)/2} \pmod 2,$$

and so $b_2(n)$ is odd if and only if $n = \ell(3\ell+1)/2$ for some $\ell \in \mathbb{Z}$. Thus, in contrast to the case of $p(n)$ we have a complete answer for the 2-divisibility of $b_2(n)$ (see [6] and [3] for analogous results for $k \in \{3, 5, 7, 11\}$).

Now consider the m-divisibility of $b_k(n)$ when $(m, k) = 1$. In [2] Ahlgren and Lovejoy prove that if $p \geq 5$ is prime, then for any $j \geq 1$ the arithmetic density of positive integers n such that $b_2(n) \equiv 0 \pmod{p^j}$ is at least $\frac{p+1}{2p}$ (they also prove that $b_2(n)$ satisfies Ramanujan-type congruences modulo p^j). In [9] Penniston extended this to show that for distinct primes k and p with $3 \leq k \leq 23$ and $p \geq 5$, the arithmetic density of positive integers n for which $b_k(n) \equiv 0 \pmod{p^j}$ is at least $\frac{p+1}{2p}$ if $p \nmid k-1$, and at least $\frac{p-1}{p}$ if $p \mid k-1$ (in [11] and [12] Treneer has shown that divisibility and congruence results such as these hold for general k). The latter result indicates that a special role may be played by the prime divisors of $k-1$, and we investigate this here. In particular, we focus on the 2-divisibility of $b_5(n)$ and the 2- and 3-divisibility of $b_{13}(n)$.

Theorem 1. *Let n be a nonnegative integer. Then $b_5(2n)$ is odd if and only if $n = \ell(3\ell+1)$ for some $\ell \in \mathbb{Z}$. That is,*

$$\sum_{n=0}^{\infty} b_5(2n)q^{2n} \equiv \sum_{\ell \in \mathbb{Z}} q^{2\ell(3\ell+1)} \pmod 2.$$

Remark. By Euler's Pentagonal Number Theorem, Theorem 1 is equivalent to

$$\sum_{n=0}^{\infty} b_5(2n)q^{2n} \equiv \prod_{n=1}^{\infty} (1-q^n)^4 \pmod 2. \qquad (2)$$

Theorem 2. *Let n be a nonnegative integer. Then $b_{13}(2n)$ is odd if and only if $n = \ell(\ell+1)$ or $n = 13\ell(\ell+1)+3$ for some nonnegative integer ℓ. That is,*

$$\sum_{n=0}^{\infty} b_{13}(2n)q^{2n} \equiv \sum_{\ell=0}^{\infty} q^{2\ell(\ell+1)} + \sum_{\ell=0}^{\infty} q^{26\ell(\ell+1)+6} \pmod 2.$$

Remark. Jacobi's triple product formula yields

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{\ell=0}^{\infty} (-1)^{\ell}(2\ell+1)q^{\ell(\ell+1)/2},$$

and hence Theorem 2 is equivalent to

$$\sum_{n=0}^{\infty} b_{13}(2n)q^{2n} \equiv \prod_{n=1}^{\infty}(1-q^{4n})^3 + q^6 \cdot \prod_{n=1}^{\infty}(1-q^{52n})^3 \pmod{2}. \qquad (3)$$

Theorems 1 and 2 yield infinitely many Ramanujan-type congruences modulo 2 for $b_5(n)$ and $b_{13}(n)$ in even arithmetic progressions. It turns out that our proof of Theorem 1 yields two congruences for $b_5(n)$ in odd arithmetic progressions.

Theorem 3. *For every* $n \geq 0$,

$$\begin{aligned} & b_5(20n+5) && \equiv 0 \pmod{2} \\ \textit{and} \quad & b_5(20n+13) && \equiv 0 \pmod{2}. \end{aligned}$$

Finally, we make the following conjecture regarding the 3-divisibility of $b_{13}(n)$.

Conjecture 1. *For any* $\ell \geq 2$,

$$b_{13}\left(3^{\ell}n + \frac{5 \cdot 3^{\ell-1}-1}{2}\right) \equiv 0 \pmod{3}$$

for every $n \geq 0$.

It turns out (see Proposition 2 below) that one can reduce the verification of each of the congruences in Conjecture 1 to a finite computation. We have verified the conjecture for each $2 \leq \ell \leq 6$.

2 Modular forms

We begin with some background on the theory of modular forms. Given a positive integer N, let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let $\mathbb{H} = \{z \mid \Im(z) > 0\}$ be the complex upper half plane, and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$ define $\gamma z := \frac{az+b}{cz+d}$.

Suppose k is a positive integer, $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and χ is a Dirichlet character modulo N. Then f is said to be a *modular form* of weight k, level N, and character χ if

$$f(\gamma z) = \chi(d)(cz+d)^k f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and f is holomorphic at the cusps of $\Gamma_0(N)$. The modular forms of weight k, level N and character χ form a finite-dimensional complex vector space which we denote by $M_k(\Gamma_0(N), \chi)$ (we will often omit χ from our notation when it is the trivial character). For instance, if we denote by $\theta(z)$ the classical theta function

$$\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots \qquad (q := e^{2\pi i z}),$$

then $\theta^4(z) \in M_2(\Gamma_0(4))$ (see, for example, [5]).

A theorem of Sturm [10] provides a method to test whether two modular forms are congruent modulo a prime. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ has integer coefficients and m is a positive integer, let $\mathrm{ord}_m(f(z))$ be the smallest n for which $a(n) \not\equiv 0 \pmod{m}$ (if there is no such n, we define $\mathrm{ord}_m(f(z)) := \infty$).

Theorem 4. *(Sturm) Suppose N is a positive integer, p is prime and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If*

$$\mathrm{ord}_p(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)],$$

then $f(z) \equiv g(z) \pmod{p}$.

We note here that $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod \left(\frac{\ell+1}{\ell}\right)$, where the product is over the prime divisors of N.

Hecke operators play a crucial role in the proofs of our results. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$, then the action of the Hecke operator $T_{p,k,\chi}$ on $f(z)$ is defined by

$$(f \mid T_{p,k,\chi})(z) := \sum_{n=0}^{\infty} (a(pn) + \chi(p)p^{k-1}a(n/p))q^n$$

(we follow the convention that $a(x) = 0$ if $x \notin \mathbb{Z}$). Notice that if $k > 1$, then

$$(f \mid T_{p,k,\chi})(z) \equiv \sum_{n=0}^{\infty} a(pn)q^n \pmod{p}. \tag{4}$$

Moreover, if $f(z) \in M_k(\Gamma_0(N), \chi)$, then $(f|T_{p,k,\chi})(z) \in M_k(\Gamma_0(N), \chi)$. When k and χ are clear from context, we will write $T_p := T_{p,k,\chi}$.

The next proposition follows directly from (4) and the definition of $T_{p,k,\chi}$.

Proposition 1. *Suppose p is prime, $g(z) \in \mathbb{Z}[[q]]$, $h(z) \in \mathbb{Z}[[q^p]]$ and $k > 1$. Then $(gh \mid T_{p,k,\chi})(z) \equiv (g \mid T_{p,k,\chi})(z) \cdot h(z/p) \pmod{p}$.*

We will construct modular forms using Dedekind's eta function, which is defined by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

for $z \in \mathbb{H}$. A function of the form

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z), \tag{5}$$

where $N \geq 1$, $r_\delta \in \mathbb{Z}$ and the product is over the positive divisors of N, is called an *eta-quotient.*

From ([8], p. 18), if $f(z)$ is the eta-quotient (5), $k := \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}$,

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$N \sum_{\delta | N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24},$$

then $f(z)$ satisfies the transformation property

$$f(\gamma z) = \chi(d)(cz + d)^k f(z)$$

for all $\gamma \in \Gamma_0(N)$. Here χ is given by $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s := \prod_{\delta | N} \delta^{r_\delta}$. Assuming that f satisfies these conditions, then since $\eta(z)$ is analytic and does not vanish on $\mathbb{H}$, we have that $f(z) \in M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) if $f(z)$ is holomorphic (resp. vanishes) at the cusps of $\Gamma_0(N)$. By ([8], Theorem 1.65) we have that if c and d are positive integers with $(c, d) = 1$ and $d \mid N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{N}{24 d (d, \frac{N}{d})} \cdot \sum_{\delta | N} \frac{(d, \delta)^2 r_\delta}{\delta}.$$

3 Proof of the main results

Proof of Theorem 1. We begin by constructing several modular forms. First let

$$f(z) := \frac{\eta^5(z)}{\eta(5z)} = 1 + 5q + 15q^2 + 40q^3 + 95q^4 + 205q^5 + \cdots.$$

Define the character χ_m by $\chi_m(d) := \left(\frac{m}{d}\right)$. Using the results on eta-quotients cited above we find that $f(z) \in M_2(\Gamma_0(5), \chi_5)$, and hence $f(2z) \in M_2(\Gamma_0(10), \chi_5)$. Next, letting

$$g(z) := \frac{\eta^5(5z)}{\eta(z)} = q - q^2 - q^4 + q^5 + 4q^6 + \cdots$$

we have that $g(z) \in M_2(\Gamma_0(5), \chi_5)$ and $g(2z) \in M_2(\Gamma_0(10), \chi_5)$. Finally, recall that

$$\theta^4(z) = 1 + 8q + 24q^2 + 32q^3 + \cdots \in M_2(\Gamma_0(4)).$$

Notice that $f^2(z)$, $f^2(2z)$, $g^2(z)$, $g^2(2z)$, and $(\theta^4(z))^2$ are all in $M_4(\Gamma_0(20))$.

From (1) we have

$$\begin{aligned} g(z) &= \frac{\eta(5z)}{\eta(z)} \cdot \eta^4(5z) \\ &= \frac{q^{5/24} \prod_{n=1}^{\infty}(1-q^{5n})}{q^{1/24} \prod_{n=1}^{\infty}(1-q^n)} \cdot q^{20/24} \prod_{j=1}^{\infty}(1-q^{5j})^4 \\ &\equiv \sum_{n=0}^{\infty} b_5(n) q^{n+1} \prod_{j=1}^{\infty}(1+q^{20j}) \pmod 2. \end{aligned}$$

It follows from Proposition 1 that

$$(g \mid T_2)(z) \equiv \sum_{n=0}^{\infty} b_5(2n+1) q^{n+1} \cdot \prod_{j=1}^{\infty}(1-q^{10j}) \pmod 2, \tag{6}$$

and hence

$$g(z) - (g \mid T_2)(2z) \equiv \sum_{n=0}^{\infty} b_5(2n) q^{2n+1} \cdot \prod_{j=1}^{\infty}(1-q^{20j}) \pmod 2.$$

We have seen that $g(2z), (g \mid T_2)(2z) \in M_2(\Gamma_0(10), \chi_5)$, and one can easily check that these two forms are congruent modulo 2 out to their q^3 terms. By Sturm's theorem we conclude that they are congruent modulo 2, and so by (3),

$$\sum_{n=0}^{\infty} b_5(2n) q^{2n+1} \cdot \prod_{j=1}^{\infty}(1+q^{20j}) \equiv g(z) + g(2z) \pmod 2. \tag{7}$$

Since

$$g(z)+g(2z)=q\cdot\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)}+q^2\cdot\prod_{n=1}^{\infty}\frac{(1-q^{10n})^5}{(1-q^{2n})} \tag{8}$$

and $1-q^{20j}\equiv(1-q^{5j})^4 \pmod 2$, (7) and (8) yield

$$\sum_{n=0}^{\infty}b_5(2n)q^{2n}\equiv\prod_{n=1}^{\infty}\frac{(1-q^{5n})}{(1-q^n)}+q\cdot\prod_{n=1}^{\infty}\frac{(1-q^{5n})^6}{(1-q^{2n})}\pmod 2. \tag{9}$$

Note that we have the following congruence of series:

$$(g(z)+\theta^4(z))^2\equiv g^2(z)+(\theta^4(z))^2\pmod 2. \tag{10}$$

Applying Sturm's theorem again, we find that

$$g^2(z)+(\theta^4(z))^2\equiv f^2(z)\pmod 2. \tag{11}$$

Since $\theta(z)\equiv 1 \pmod 2$, (10) and (11) yield

$$g(z)+1\equiv f(z)\pmod 2,$$

and therefore

$$\frac{1}{f(z)}+\frac{g(z)}{f(z)}\equiv 1\pmod 2.$$

This implies that

$$1\equiv\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^4}\cdot\left(\prod_{n=1}^{\infty}\frac{(1-q^{5n})}{(1-q^n)}+q\cdot\prod_{n=1}^{\infty}\frac{(1-q^{5n})^6}{(1-q^{2n})}\right)\pmod 2, \tag{12}$$

and (2) now follows from (9) and (12). $\square$

Proof of Theorem 2. To begin, we define several modular forms in the space $M_6(\Gamma_0(13),\chi_{13})$:

$$w_\ell(z):=\eta^{2\ell-3}(13z)\eta^{15-2\ell}(z)\qquad(1\leq\ell\leq 8).$$

We also need

$$\theta^{12}(z)\in M_6(\Gamma_0(4)).$$

Notice that $w_\ell(z)^2,(\theta^{12}(z))^2\in M_{12}(\Gamma_0(52))$.

From (1) we have that

$$w_8(z)\equiv\sum_{n=0}^{\infty}b_{13}(n)q^{n+7}\cdot\prod_{j=1}^{\infty}(1-q^{52j})^3\pmod 2.$$

Then

$$(w_8 \mid T_2)(z) \equiv \sum_{n=0}^{\infty} b_{13}(2n+1) q^{n+4} \cdot \prod_{j=1}^{\infty} (1-q^{26j})^3 \pmod{2},$$

and hence

$$w_8(z) - (w_8 \mid T_2)(2z) \equiv \sum_{n=0}^{\infty} b_{13}(2n) q^{2n+7} \cdot \prod_{j=1}^{\infty} (1-q^{52j})^3 \pmod{2}. \tag{13}$$

By Sturm's theorem we obtain

$$(w_8 \mid T_2)(z) \equiv w_5(z) + w_8(z) \pmod{2},$$

and combining this with (13) we find that

$$\sum_{n=0}^{\infty} b_{13}(2n) q^{2n+7} \cdot \prod_{j=1}^{\infty} (1-q^{13j})^{12} \equiv q^7 \cdot \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{13}}{(1-q^n)}$$
$$+q^8 \cdot \prod_{n=1}^{\infty} (1-q^{13n})^{14} (1-q^n)^{10} + q^{14} \cdot \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{26}}{(1-q^n)^2} \pmod{2}.$$

From this we see that

$$\sum_{n=0}^{\infty} b_{13}(2n) q^{2n} \equiv \prod_{n=1}^{\infty} \frac{(1-q^{13n})}{(1-q^n)} + q \cdot \prod_{n=1}^{\infty} (1-q^{13n})^2 (1-q^n)^{10}$$
$$+q^7 \cdot \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{14}}{(1-q^n)^2} \pmod{2}. \tag{14}$$

Sturm's theorem provides the congruence

$$(\theta^{12}(z))^2 \equiv w_1^2(z) + w_2^2(z) + w_7^2(z) + w_8^2(z) \pmod{2},$$

and therefore

$$\theta^{12}(z) \equiv w_1(z) + w_2(z) + w_7(z) + w_8(z) \pmod{2}.$$

From this we obtain

$$1 \equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^{13}}{(1-q^{13n})} + q \cdot \prod_{n=1}^{\infty} (1-q^{13n})(1-q^n)^{11}$$

$$+q^6 \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{11}(1-q^n) + q^7 \cdot \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{13}}{(1-q^n)} \pmod{2},$$

which yields

$$\prod_{n=1}^{\infty} \frac{(1-q^{13n})}{(1-q^n)} \equiv \prod_{n=1}^{\infty}(1-q^n)^{12} + q \cdot \prod_{n=1}^{\infty}(1-q^{13n})^2(1-q^n)^{10}$$

$$+q^6 \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{12} + q^7 \cdot \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{14}}{(1-q^n)^2}.$$

Combining this with (14) we find that

$$\begin{aligned} \sum_{n=0}^{\infty} b_{13}(2n)q^{2n} &\equiv \prod_{n=1}^{\infty}(1-q^n)^{12} + q^6 \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{12} \\ &\equiv \prod_{n=1}^{\infty}(1-q^{4n})^3 + q^6 \cdot \prod_{n=1}^{\infty}(1-q^{52n})^3 \pmod{2}, \end{aligned}$$

which is (3). □

Proof of Theorem 3. We prove only the first congruence, as the second can be proved in a similar way. We saw above that $g(2z)$ and $(g|T_2)(2z)$ are congruent modulo 2. The same is then true of $g(z)$ and $(g|T_2)(z)$, which by (6) gives

$$\sum_{n=0}^{\infty} b_5(2n+1)q^{n+1} \cdot \prod_{j=1}^{\infty}(1-q^{10j}) \equiv q \cdot \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)} \pmod{2}.$$

Then

$$\sum_{n=0}^{\infty} b_5(2n+1)q^{n+1} \cdot \prod_{j=1}^{\infty}(1-q^{10j}) \equiv \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^n)} \cdot q \cdot \prod_{n=1}^{\infty}(1-q^{5n})^4 \pmod{2},$$

and hence

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv \sum_{\ell=0}^{\infty} b_5(\ell)q^{\ell} \cdot \prod_{j=1}^{\infty}(1-q^{10j}) \pmod{2}. \tag{15}$$

Note that $2n+1$ has the form $20m+5$ if and only if $n \equiv 2 \pmod{10}$. Since the infinite product on the right hand side of (15) only produces powers of q that are 0 modulo 10, it suffices to show that

$$b_5(10n+2) \equiv 0 \pmod{2} \tag{16}$$

for all $n \geq 0$. One can easily check that the congruence $6\ell^2 + 2\ell \equiv 2 \pmod{10}$ has no solution, and so (16) follows from Theorem 1.

□

With regard to Conjecture 1, we have the following elementary proposition.

Proposition 2. *Let $\ell \geq 2$. If the congruence*

$$b_{13}\left(3^\ell n + \frac{5 \cdot 3^{\ell-1} - 1}{2}\right) \equiv 0 \pmod{3}$$

holds for all $0 \leq n \leq 7 \cdot 3^{\ell-1} - 3$, then it holds for all $n \geq 0$.

Proof. The idea of our proof is to repeatedly apply the T_3 operator to the modular form

$$P_\ell(z) := \frac{\eta(13z)}{\eta(z)} \cdot \eta^e(13z),$$

where $e := 4 \cdot 3^\ell$. By the criteria for eta-quotients cited above, $P_\ell(z) \in M_{\frac{e}{2}}(\Gamma_0(13), \chi_{13})$.

For each $1 \leq s \leq \ell$ let

$$\delta_s := \frac{13 \cdot 3^{s-1} + 1}{2}.$$

Then

$$P_\ell(z) = \sum_{n=0}^{\infty} b_{13}(n) q^{n+\delta_\ell} \cdot \prod_{j=1}^{\infty} (1 - q^{13j})^e.$$

Using Proposition 1 and the fact that $\delta_s \equiv 2 \pmod{3}$ for $2 \leq s \leq \ell$, an easy induction argument gives that $(P_\ell \mid T_3^s)(z)$ is congruent modulo 3 to

$$\sum_{n=0}^{\infty} b_{13}\left(3^s n + \left(\frac{3^s - 1}{2}\right)\right) q^{n+\delta_{\ell-s}} \cdot \prod_{j=1}^{\infty} (1 - q^{52j})^{3^{\ell-s}} \pmod{3}.$$

for any $1 \leq s \leq \ell - 1$. In particular,

$$(P_\ell \mid T_3^{\ell-1})(z) \equiv \sum_{n=0}^{\infty} b_{13}\left(3^{\ell-1} n + \left(\frac{3^{\ell-1} - 1}{2}\right)\right) q^{n+7} \cdot \prod_{j=1}^{\infty} (1 - q^{156j}) \pmod{3}.$$

Then

$$(P_\ell|T_3^\ell)(z) \equiv \sum_{n=0}^{\infty} b_{13}\left(3^{\ell-1}(3n+2)+\left(\frac{3^{\ell-1}-1}{2}\right)\right) q^{\frac{(3n+2)+7}{3}} \cdot \prod_{j=1}^{\infty}(1-q^{52j})$$
$$\equiv \sum_{n=0}^{\infty} b_{13}\left(3^\ell n+\frac{5\cdot 3^{\ell-1}-1}{2}\right) q^{n+3} \cdot \prod_{j=1}^{\infty}(1-q^{52j}) \pmod 3.$$

Since $(P_\ell \mid T_3^\ell)(z) \in M_{\frac{c}{2}}(\Gamma_0(13), \chi_{13})$, by Sturm's theorem we have that if $ord_3((P_\ell \mid T_3^\ell)(z)) > 7\cdot 3^{\ell-1}$, then $(P_\ell \mid T_3^\ell)(z) \equiv 0 \pmod 3$. Therefore, if the congruence

$$b_{13}\left(3^\ell n+\frac{5\cdot 3^{\ell-1}-1}{2}\right) \equiv 0 \pmod 3$$

holds for all $0 \leq n \leq 7\cdot 3^{\ell-1}-3$, then it holds for all $n \geq 0$. □

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