A NOTE ON THE IRREDUCIBILITY OF HECKE POLYNOMIALS

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Let S_{2k} denote the space of modular cusp forms of weight 2k for $SL_2(\mathbb{Z})$. If $f(z) := \sum_{n=1}^{\infty} a_f(n)q^n \in S_{2k}$ (here $q := e^{2\pi i z}$ throughout), and p is prime, then the Hecke operator T_p^{2k} is the linear transformation on S_{2k} given by

$$T_p^{2k}f := \sum_{n=1}^{\infty} \left(a_f(np) + p^{2k-1}a_f(n/p) \right) q^n.$$

Let $T_p^{2k}(x)$ denote the characteristic polynomial of the action of T_p^{2k} on S_{2k} . It is well known that $T_p^{2k}(x) \in \mathbb{Z}[x]$ and has degree d_k where $d_k := \dim(S_{2k})$. However, much more is conjectured to be true. Maeda has conjectured that the Hecke algebra of S_{2k} over \mathbb{Q} is simple, and that its Galois closure over \mathbb{Q} has Galois group S_{d_k} . Recently there have been numerous investigations regarding the irreducibility of characteristic polynomials of Hecke operators on S_{2k} . The existence of such polynomials have proven to be useful in proving nonvanishing theorems for central values of level 1 modular *L*-functions, and in constructing base changes to totally real number fields for level 1 eigenforms (see [C-F,H-M,Ko-Z]).

In this note we show that a "positive proportion" of the Hecke polynomials $T_p^{2k}(x)$ are irreducible if there are two distinct primes ℓ and q for which $T_q^{2k}(x)$ is irreducible over \mathbb{F}_{ℓ} , the finite field with ℓ elements. Throughout p will denote a prime number.

Theorem 1. If there are distinct primes ℓ and q for which the Hecke polynomial $T_q^{2k}(x)$ is irreducible in $\mathbb{F}_{\ell}[x]$, then

$$\#\left\{p < X \mid T_p^{2k}(x) \text{ is irreducible in } \mathbb{Q}[x]\right\} \gg_k \frac{X}{\log X}.$$

This result follows immediately from the following more general result.

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As usual, we will let $S_k(N,\chi)$ denote the space of modular cusp forms of weight k, level N and character χ . For $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N,\chi)$ and $p \nmid N$ the Hecke operator $T_{N,p}^{k,\chi}$ is defined by

$$T_{N,p}^{k,\chi}f = \sum_{n \ge 1} \left(a_f(np) + \chi(p)p^{k-1}a_f(n/p) \right) q^n.$$

Let $T_{N,p}^{k,\chi}(x)$ denote the characteristic polynomial of $T_{N,p}^{k,\chi}$ on $S_k^{\text{new}}(N,\chi)$. Moreover, let $K_{k,\chi,N}$ denote the finite extension of \mathbb{Q} obtained by adjoining the roots of all of the $T_{N,p}^{k,\chi}(x)$ with $p \nmid N$.

Theorem 2. Let q and ℓ be distinct primes not dividing N, and let \mathcal{L} denote a prime ideal of $\mathbb{K}_{k,\chi,N}$ lying above ℓ . Then

$$\#\left\{p < X \mid T_{N,p}^{k,\chi}(x) \equiv T_{N,q}^{k,\chi}(x) \pmod{\mathcal{L}}\right\} \gg_{N,\chi,k} \frac{X}{\log X}.$$

Proof. Let $\{f_1, \ldots, f_d\}$ be a basis of $S_k^{\text{new}}(N, \chi)$ such that each f_i is an eigenform for all of the $T_{N,p}^{k,\chi}$ where $p \nmid N$. Let $\lambda_{f_i}(p)$ be the eigenvalue of $T_{N,p}^{k,\chi}$ corresponding to the eigenform f_i (i.e. $T_{N,p}^{k,\chi}f_i = \lambda_{f_i}(p)f_i$). By the work of Deligne, Serre and Shimura [D,D-S,Sh], there exist continuous representations

$$\rho_{f_i,\mathcal{L}}: \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathcal{O}_{\mathcal{L}})$$

satisfying the following conditions:

- (i) $\rho_{f_i,\mathcal{L}}$ is unramified for $p \nmid N\ell$,
- (ii) trace $(\rho_{f_i,\mathcal{L}}(\operatorname{Frob}_p)) = \lambda_{f_i}(p)$ for $p \nmid N\ell$,
- (iii) det $(\rho_{f_i,\mathcal{L}}(\operatorname{Frob}_p)) = \chi(p)p^{k-1}$ for $p \nmid N\ell$.

Here \mathcal{O} denotes the ring of integers of $K_{k,\chi,N}$, $\mathcal{O}_{\mathcal{L}}$ denotes its completion at \mathcal{L} and Frob_p denotes any Frobenius element for p. Let \mathfrak{l} denote a uniformizer for $\mathcal{O}_{\mathcal{L}}$. By reducing the representations $\rho_{f_i,\mathcal{L}}$ modulo \mathfrak{l} , we obtain representations from $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\operatorname{GL}_2(\mathcal{O}_{\mathcal{L}}/\langle \mathfrak{l} \rangle)$. These representations have finite image and therefore factor to yield representations:

$$\bar{\rho}_{f_i,\mathcal{L}} : \operatorname{Gal}(K/\mathbb{Q}) \to \operatorname{GL}_2(\mathcal{O}_{\mathcal{L}}/\langle \mathfrak{l} \rangle),$$

where K is a finite extension of \mathbb{Q} .

By the Chebotarev density theorem, there is a set S of rational primes having positive Dirichlet density and having the property that for each $p \in S$, $\operatorname{Frob}_{\mathfrak{P}}$ is conjugate to $\operatorname{Frob}_{\mathfrak{Q}}$, where \mathfrak{P} and \mathfrak{Q} are prime ideals of K lying above p and q respectively, and therefore have the property that for each $1 \leq i \leq d$

$$\operatorname{trace}(\bar{\rho}_{f_i,\mathcal{L}}(\operatorname{Frob}_{\mathfrak{P}})) \equiv \lambda_{f_i}(p) \pmod{\mathfrak{l}} \quad \text{if } p \nmid N\ell$$

Since the trace is conjugation invariant, it follows that for $p \in S$,

$$\lambda_{f_i}(p) \equiv \lambda_{f_i}(q) \pmod{\mathfrak{l}}.$$

Since the $\lambda_{f_i}(p)$ $(1 \le i \le d)$ are precisely the roots of $T^{k,\chi}_{N,p}(x)$, the theorem follows. \Box

Example. Here we shall give a simple example that illustrates Theorem 2. We consider the Hecke polynomials on the three dimensional space $S_5^{\text{new}}(11, \chi_{-11})$. For convenience, let $T_p(x)$ denote the characteristic polynomial for the Hecke operator $T_{11,p}^{5,\chi_{-11}}$. The first few terms of the Fourier expansions of the three newforms are

$$N_{1}(z) = \sum_{n=1}^{\infty} a_{1}(n)q^{n} = q + 7q^{3} + 16q^{4} - 49q^{5} - 32q^{9} + \cdots,$$

$$N_{2}(z) = \sum_{n=1}^{\infty} a_{2}(n)q^{n} = q + \sqrt{-30}q^{2} - 3q^{3} - 14q^{4} + 31q^{5} - 3\sqrt{-30}q^{6} - 10\sqrt{-30}q^{7} + \cdots,$$

$$N_{3}(z) = \sum_{n=1}^{\infty} a_{3}(n)q^{n} = q - \sqrt{-30}q^{2} - 3q^{3} - 14q^{4} + 31q^{5} + 3\sqrt{-30}q^{6} + 10\sqrt{-30}q^{7} - \cdots.$$

It is easy to verify that if $p \neq 11$ is prime, then

(1)
$$a_2(p) = a_3(p) \in \mathbb{Z}$$
 if $\left(\frac{p}{11}\right) = 1$

and

(2)
$$a_2(p) = -a_3(p)$$
 if $\left(\frac{p}{11}\right) = -1$.

Moreover, if $\left(\frac{p}{11}\right) = -1$, then

(3)
$$\frac{a_2(p)}{\sqrt{-30}} \in \mathbb{Z}.$$

These all follow from standard facts about eigenvalues of Hecke operators (e.g. [Kob]).

The form $N_1(z)$ has complex multiplication by $\mathbb{Q}(\sqrt{-11})$ in the sense of Ribet (see [R]). By construction, there is exactly one such form in this space. In particular if $p \neq 11$ is prime, then

$$a_1(p) = \begin{cases} 0 & \text{if } \left(\frac{p}{11}\right) = -1, \\ \frac{2x^4 - 132x^2y^2 + 242y^4}{16} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ and } 4p = x^2 + 11y^2. \end{cases}$$

This implies that if $p \neq 11$ is prime, then

(4)
$$a_1(p) \equiv \begin{cases} 0 \pmod{11} & \text{if } \left(\frac{p}{11}\right) = -1, \\ 2p^2 \pmod{11} & \text{if } \left(\frac{p}{11}\right) = 1. \end{cases}$$

Now if $B(z) = \sum_{n=1}^{\infty} b(n)q^n$ is defined by

$$B(z) := \frac{15 + \sqrt{-30}}{30} \cdot N_2(z) + \frac{15 - \sqrt{-30}}{30} \cdot N_3(z) = q - 2q^2 - 3q^3 - \cdots,$$

then the methods of Swinnerton-Dyer [S-D] suggest that B(z) may satisfy a congruence with a linear combination of twisted Eisenstein series. Using a theorem of Sturm [St], we verify indeed that there is such a congruence modulo 11, and it turns out that

(5)
$$b(n) \equiv \left(8n + 4n\left(\frac{n}{11}\right)\right) \sum_{d|n} d^7 \pmod{11}.$$

By combining (1-5), if $p \neq 11$ is prime, then

$$T_p(x) \equiv \begin{cases} x^3 + 5x^2 + x + 3 \pmod{11} & \text{if } p \equiv 1 \pmod{11}, \\ x^3 + 8x \pmod{11} & \text{if } p \equiv 2 \pmod{11}, \\ x^3 + 10x^2 + 3 \pmod{11} & \text{if } p \equiv 3 \pmod{11}, \\ x^3 + 8x^2 + 4 \pmod{11} & \text{if } p \equiv 4 \pmod{11}, \\ x^3 + 9x^2 + 2x + 9 \pmod{11} & \text{if } p \equiv 5 \pmod{11}, \\ x^3 + 10x \pmod{11} & \text{if } p \equiv 6 \pmod{11}, \\ x^3 + 8x \pmod{11} & \text{if } p \equiv 7 \pmod{11}, \\ x^3 + 7x \pmod{11} & \text{if } p \equiv 8 \pmod{11}, \\ x^3 + x^2 + 6x + 3 \pmod{11} & \text{if } p \equiv 9 \pmod{11}, \\ x^3 \pmod{11} & \text{if } p \equiv 10 \pmod{11}. \end{cases}$$

If p is a prime for which $\left(\frac{p}{11}\right) = 1$, then by (1) and (4) we see that $T_p(x)$ factors into three linear factors in $\mathbb{Z}[x]$. If p is a prime for which $\left(\frac{p}{11}\right) = -1$ and $a_2(p) \neq 0$, then by (2) and (3) it follows that $T_p(x)$ factors into irreducibles in $\mathbb{Z}[x]$ as

$$T_p(x) = x(x^2 + a_2(p)^2).$$

By (5) one easily finds that $a_2(p) \neq 0$ for every such $p \equiv 2, 6, 7, 8 \pmod{11}$.

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