Counting Kings: As Easy As λ_1, λ_2, λ_3 ...

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Abstract

Let $F(m, n)$ be the number of distinct configurations of non-attacking kings on an $m \times n$ chessboard. Let $\eta_2 = lim_{m,n\to\infty} F(m, n)^{\frac{1}{mn}}$. We give rigorous and heuristic bounds for η_2. We also give bounds for similar constants in higher dimensions.

1 Introduction

We consider the following question: How many different ways can kings be placed on a chessboard so that no two kings can attack each other? How about on an $m \times n$ chessboard? The statement of the problem generalizes naturally to a d-dimensional board.

In chess a king can attack any of the 8 squares surrounding the square in which the king is placed. Kings in the center of the board can attack any of the 8 surrounding squares while kings on the boundary of the board attack fewer.

In this paper, we will first examine the Kings Problem in one dimension, then discuss the problem in two dimensions, and will eventually approach the problem in higher dimensions. Our primary approach will utilize adjacency matrices and the dominant eigenvalues of these matrices. We will attempt to bound the entropy constant of each system,η_d, where d is the number of dimensions of the board.

2 The One-Dimensional Problem

A one dimensional chess board is simply a row or column of squares. Squares at either end are adjacent to exactly one other square and all other

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squares are adjacent to two. A king in any square can attack any adjacent square.

Definition 1. $F(n)$ = *the number of distinct configurations of non-attacking kings on a one-dimensional chess board with* n *squares.*

Consider $F(n)$. Every configuration on a board of length n will either begin with an empty square or begin with a square with a king in it. Those that begin with an empty square are followed by a board of $n-1$ squares. Since the begining square is empty, every configuration of kings on a board of $n-1$ squares can follow the initial empty square. Similarly, configurations with a king in the first square must have an empty square second and are then followed by a board of $n-2$ squares. Since the second square is empty, every configuration of kings on a board of $n-2$ squares can follow. Thus $F(n) = F(n-1) + F(n-2)$ for $n > 2$.

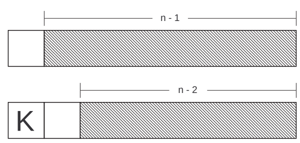

Figure 1: Configurations of non-attacking kings on 1 by n boards

Since it is easy to determine that $F(1) = 2$ and $F(2) = 3$ we have a complete recursive definition of $F(n)$:

$$F(n) = \begin{cases} 2 & n = 1 \\ 3 & n = 2 \\ F(n-1) + F(n-2) & n > 2 \end{cases}$$

It is interesting to note that $F(n)$ gives the familiar Fibonacci sequence properly indexed.

3 The Two-Dimensional Problem

Definition 2. $F(m, n)$ = *the number of distinct configurations of non-attacking kings on an* m *by* n *chess board. It is obvious that* $F(m, n) = F(n, m)$

In order to thoroughly study the kings problem in two dimensions it will be useful to define another object in the one dimensional kings problem.

Definition 3. S_n = *the set of all possible configurations of non-attacking kings on a one dimensional chess board with* n *squares.*

Clearly $\#S_n = F(n)$. There is a method of generating the elements of S_n in an order that will be useful later.

1. Write the numbers 0 through F(n)-1 as a sum of the fewest non-repeating Fibonacci numbers. Thus $1 = 1$, $2 = 2$, $3 = 3$, $4 = 3 + 1$.

2. Create a grid with rows labeled $0, 1, 2, \dots, F(n) - 1$ and columns labeled with Fibonacci numbers beginning with $1, 2$ up to $F(n - 1)$

3. Place a king in each square that appears in the sum. The zeroth row of squares has no king, The first row has a king in the first square, the second row of kings has a king in the second square, the forth row of squares has kings in the 1st and 3rd square.

It should be clear that this process will enumerate every possible one dimensional board, because no two consecutive Fibonacci numbers will appear in a sum, so no two kings will be placed in adjacent squares. The following diagram may aid the visualization.

	1	2	3	5	8
0					
1	K				
2		K			
3			K		
4	K		K		
5				K	
6	K			K	
7		K		K	
8					K
9	K				K
10		K			K
11			K		K
12	K		K		K

Figure 2: The elements of S_5 generated by summing Fibonacci numbers

One result of generating S_5 in this order is that we have also generated the elements of $S_n \forall n \leq 5$. For example S_4 is the first 8 rows of the grid with the last column removed.

Using methods from Biggs[2] we use the approach of graphs and adjacency matrices to bring us from the one-dimensional problem to the two dimensional problem. We construct a graph G_n whose vertices are the elements of S_n. Two vertices are adjacent if and only if the two 1-dimensional boards can be glued together as a permissible 2 dimensional board. Thus the number of walks on the graph of length $m-1$ will correspond to the number of legal configurations on an $m \times n$ board. We let A_n be the adjacency matrix associated with G_n. Thus,

$$F(m,n) = \mathbf{1}^T A_n^{m-1} \mathbf{1}$$

The construction of the set S_n provides a convenient recurrence in the matrices A_n, illustrated below.

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And in general:

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-2} \\ A_{n-2} & 0 \end{pmatrix}$$

Where the copies of A_{n-2} are padded with rows or columns of zeros as needed. Thus, we can generate A_n for any n limited only by the space in we have in which to record the matrix. We can also answer the first question we asked by finding the numbers of ways can kings be placed on a standard 8×8 chessboard so that no two kings can attack each other.

$$F(8,8) = \mathbf{1}^T A_8^7 \mathbf{1} = 1355115601$$

This recurrence gives rise to matrices with very interesting structure. Pictured below is the matrix A_{12} with values of 1 represented by a black pixel and values of 0 represented by a white pixel. We were able to use the structure to our advantage in performing certain calculations which will be discussed in the next section.

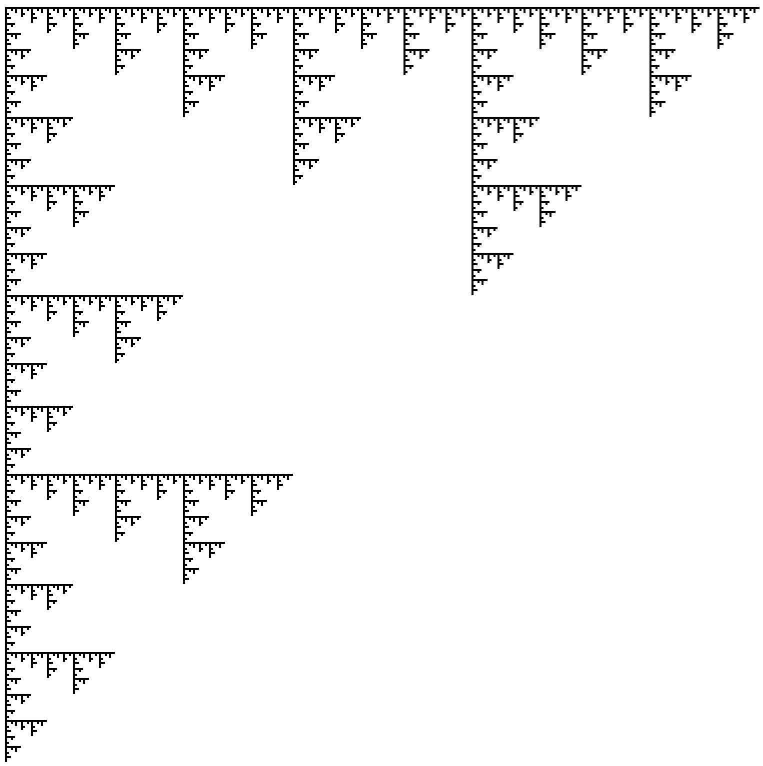

Figure 3: A_{12}

The eigenvectors associated with the dominant eigenvalues also have an interesting structure. The following are plots of the entries in $\boldsymbol{v}_9$, $\boldsymbol{v}_{10}$, and $\boldsymbol{v}_{11}$ against their indices. The vectors are normalized so that the largest entry is 1 and we have connected the values with a line in order to better see the "shape" of the vector. At first glance it appears that $\boldsymbol{v}_{11}$ is a concatenation of $\boldsymbol{v}_{10}$ and $\boldsymbol{v}_9$ scaled appropriately, but this is not the case. We can however, get very good approximations for $\boldsymbol{v}_n$ concatenating scaled copies previous eigenvectors.

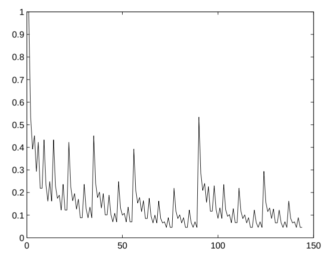

Figure 4: v_{10}

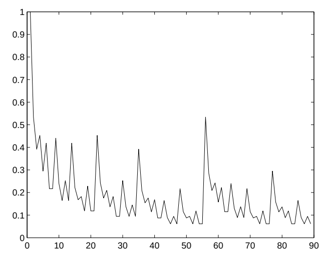

Figure 5: v_9

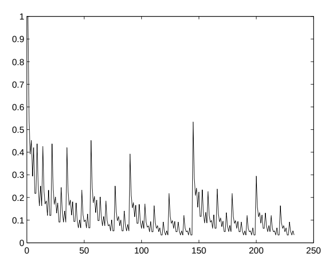

Figure 6: The actual plot of eigenvector v_{11}

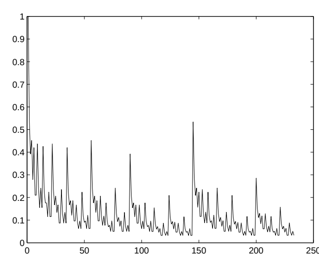

Figure 7: A concatenation of v_{10} and a scaled v_9

4 Entropy

Let $\mathbf{n} = [n_1, n_2, \ldots, n_d]$ be a vector of dimensions for a multidimensional chessboard. Let $F(\mathbf{n})$ be the number of configurations of non-attacking kings on the multidimensional board. We define the entropy constant of this system as follows:

Definition 4. $\eta_d = \lim_{\boldsymbol{n}\to\infty} F(\boldsymbol{n})^{\frac{1}{|\boldsymbol{n}|}}$, *where* $|\boldsymbol{n}| = n_1 \times n_2 ... n_d$

It is easy to see that $\eta_1 = lim_{n\to\infty} F(n)^{\frac{1}{n}}$ is the golden ratio $\frac{1+\sqrt{5}}{2}$. No closed form has been found for the entropy constants of higher dimensional systems.

In order to find bounds for η_2 we will need to work with the function $F(m,n)$. Using spectral decomposition as found in Axler[1] we have:

$$F(m,n) = \sum_i k_{n,i}\lambda_{n,i}^{m-1}$$

where each $\lambda_{n,i}$ is an eigenvalue of A_n and $k_{n,i} = \frac{\mathbf{1}^T v_{n,i} v_{n,i}^T \mathbf{1}}{||v_{n,i}||^2}$ with $v_{n,i}$ the eigenvector associated with $\lambda_{n,i}$. It is important to note that each $k_{n,i} \geq 0$ since the numerator of our expression is the square of the sum of the elements of $v_{n,i}$ and the denominator is the sum of the squares of those elements and each element is a real number since A_n is symmetric. Furthermore since each A_n is a primitive matrix we know from the Perron-Frobenius theorem that the dominant eigenvalue is simple and positive so we can order our eigenvalues so that $\lambda_{n,1} > |\lambda_{n,2}| \geq |\lambda_{n,3}| \geq \ldots$.

Theorem 1.

$$\lim_{m\to\infty} F(m,n)^{1/m} = \lambda_{n,1}.$$

Proof. We have:

$$F(m,n) = \sum_i k_{n,i}\lambda_{n,i}^{m-1}.$$

Factoring out $\lambda_{n,1}^m$ gives:

$$F(m,n) = \lambda_{n,1}^m \left(\frac{k_{n,1}}{\lambda_{n,1}} + \sum_{i=2} \frac{k_{n,i}}{\lambda_{n,1}} \left(\frac{\lambda_{n,i}}{\lambda_{n,1}} \right)^{m-1} \right).$$

So:

$$\lim_{m\to\infty} F(m,n)^{1/m} = \lambda_{n,1} \lim_{m\to\infty} \left(\frac{k_{n,1}}{\lambda_{n,1}} + \sum_{i=2} \frac{k_{n,i}}{\lambda_{n,1}} \left(\frac{\lambda_{n,i}}{\lambda_{n,1}} \right)^{m-1} \right)^{\frac{1}{m}} = \lambda_{n,1}$$

since $\frac{\lambda_{n,i}}{\lambda_{n,1}} < 1 \ \forall i > 1$. □

From this theorem and our definition of η_2 it follows that $\lim_{n\to\infty} \lambda_{n,1}^{\frac{1}{n}} = \eta_2$.

In order to develop our best lower bound on η_2, we need the following lemma.

Lemma 2. $\lambda_{n,1} F(n, 2p-1) > F(n, 2p)$

Proof.

$$\lambda_{n,1}F(n,2p-1)=\lambda_{n,1}\sum_i k_{n,1}\lambda_{n,1}^{2p-2}=\sum_i \lambda_{n,1}k_{n,i}\lambda_{n,i}^{2p-2}$$

Since each $k_{n,i}\geq 0$, $2p-2$ is even, and $\lambda_{n,1}$ is positive every term in this sum is positive. And since $\lambda_{n,1}\geq|\lambda_{n,i}|$ for all i, $\lambda_{n,1}k_{n,i}\lambda_{n,i}^{2p-2}\geq\lambda_{n,i}c_{n,i}\lambda_{n,i}^{2p-2}\forall i$. So:

$$\begin{aligned} & \lambda_{n,1}F(n,2p-1) \\ = & \textstyle\sum_i \lambda_{n,1}k_{n,i}\lambda_{n,i}^{2p-2} \\ > & \textstyle\sum_i k_{n,i}\lambda_{n,i}^{2p-1} \\ = & F(n,2p) \end{aligned}$$

Thus, $\lambda_{n,1}F(n,2p-1)>F(n,2p)$. □

Using this lemma we can prove our best lower bound. However, from this point on we are only concerned with the dominant eigenvalue of A_n so we will abbreviate our notation from $\lambda_{n,1}$ to the slightly less cumbersome λ_n.

Theorem 3. $\frac{\lambda_{2p}}{\lambda_{2p-1}}\leq\eta_2$

Proof.

$$\begin{aligned} & F(n,2p)<\lambda_n F(n,2p-1) \\ \Rightarrow & \tfrac{F(n,2p)}{F(n,2p-1)}<\lambda_n \\ \Rightarrow & \left(\tfrac{F(n,2p)}{F(n,2p-1)}\right)^{\frac{1}{n}}<\lambda_n^{\frac{1}{n}} \\ \Rightarrow & \lim_{n\to\infty}\left(\tfrac{F(n,2p)}{F(n,2p-1)}\right)^{\frac{1}{n}}\leq\lim_{n\to\infty}\lambda_n^{\frac{1}{n}} \\ \Rightarrow & \tfrac{\lambda_{2p}}{\lambda_{2p-1}}\leq\eta_2 \end{aligned}$$

□

It is clear from extensive calculation that a similar convergence is happening from above.

Conjecture 1. $\eta_2\leq\left(\frac{\lambda_{2q+1}}{\lambda_{2q}}\right)$

The best proved upper bounds come from an adaptation of the method of Calkin and Wilf in [3]. We use the fact that for each positive integer p,

$$\lambda_n\leq Trace(A_m^{2p})^{\frac{1}{2p}}.$$

We consider all the set of one-dimensional cylindrical chessboards with circumference $2p$, containing no adjacent kings. We then compute the adjacency matrices, as done before, calling them B_{2p}. Then

$$Trace(A_m^{2p})=\mathbf{1}^T B_{2p}^{m-1}\mathbf{1}.$$

So,

$$\eta_2 \le \left(\mathbf{1}^T B_{2p}^{m-1} \mathbf{1}\right)^{\frac{1}{pm}} \le \mu_{2p}^{\frac{1}{2p}}$$

where μ_{2p} is the dominant eigenvalue of B_{2p}. Since B_{2p} consists of A_{2p} with some elements zeroed out, it should be clear that $\mu_{2p} \le \lambda_{2p}$.

Our lower bound depends on our ability to calculate the dominant eigenvalue of A_n. When n is small we can use various mathematical to utilities to compute the entire eigenstructure of A_n with little difficulty. However, the size of A_n grows exponentially so we must resort to other methods.

The method we have employed is the power method with which we were able to calculate the dominant eigenvalues of matrices as large as A_{21}. We were stopped at this point due to the fact that the matrices exceeded the amount of computer memory available to us since the number of elements in A_n grows exponentially as the square of the golden ratio.

We used sparse matrix techniques to get the next few eigenvalues, but even these were overwhelmed by the rapid growth of the matrices. Since the number of non-zero elements in A_n grows like a power of 2. This is less than the rate of growth of the elements in the entire matrix, but still rapid enough to cause problems in a short amount of time.

Fortunately, the matrices we are dealing with are recursive so we developed a simple recursive algorithm to generate a single row of the matrix of interest while only storing much smaller matrices in the computer's memory. This algorithm combines sparse matrix notation with the recursive definition of A_n given earlier. As a result of this we were able to calculate the dominant eigenvalue of every matrix up to A_{34}. We have included these values in the table below.

Currently, our best proved bounds on η_2 are

$$1.3426439 \le \eta_2 \le 1.3426444$$

If proved, our conjecture would improve these bounds to

$$1.3426439509 \le \eta_2 \le 1.3426439513$$

These figures are limited only by our ability to calculate the dominant eigenvalues of these massive matrices.

We now consider chessboards of more than two dimensions. For a board drawn in d dimensions, recall that η_d is the entropy constant for the system. In order to understand such a system, it is helpful to think of the "board" as a vector space, with each square represented as a vector of integers. Two kings are adjacent if and only if the distance between them is no greater than one in each dimension. In a board of d dimensions, a centrally placed king is adjacent to $3^d - 1$ squares.

As we go forward, emphasis will be placed on η_3, because it is easiest to conceptualize, and has the most connections to real-world problems. In order to develop bounds for η_d, we will we need a supporting lemma.

n	λ_n	$\lambda_n^{\frac{1}{n}}$
1	1.6180339887499	1.618033988750
2	2.0000000000000	1.414213562373
3	2.8136065026483	1.411739131737
4	3.6903064458833	1.386007588093
5	5.0175972233734	1.380699473582
6	6.6929993658732	1.372787967659
7	9.0174511749115	1.369116936204
8	12.085328018303	1.365470447645
9	16.241777083101	1.363059576611
10	21.796006986300	1.360935973337
11	29.271986048056	1.359295363082
12	39.296415956952	1.357884231820
13	52.764934519259	1.356713379686
14	70.841810771015	1.355699845892
15	95.117240999363	1.354827332649
16	127.70723870093	1.354061748445
17	171.46630404741	1.353387875785
18	230.21752372612	1.352788522478
19	309.10064010556	1.352252801038
20	415.01176989206	1.351770674864
21	557.21327880429	1.351334692797
22	748.13887149248	1.350938427754
23	1004.4842481176	1.350576741997
24	1348.6646166621	1.350245271728
25	1810.7764482868	1.349940396006
26	2431.2280037527	1.349659030798
27	3264.2736022465	1.349398561045
28	4382.7571862615	1.349156740627
29	5884.4824399280	1.348931636768
30	7900.7647432022	1.348721573550
31	10607.913998964	1.348525092499
32	14242.651559646	1.348340917465
33	19122.809968143	1.348167927490
34	25675.125129685	1.348005133658

Lemma 4. *Let D represent a permutation of d dimensions, and $k, n \in \mathbb{Z}^+$. The following inequalities hold for boards containing $d+1$ dimensions.*

$$F(k \times n, D) < F(n, D)^k < F(k \times (n+1), D)$$

Proof. Consider the first term of the inequality. It is one big board, with the first dimension being $k \times n$ squares long. The second term represents the action of taking that board and dividing it into k equal partitions along the first dimension. This will allow additional configurations, as kings can now be placed next to each other on the newly formed edges without attacking each other. The final term in the inequality reunites the pieces of the board, with a gap of width one between them. This will have at least as many configurations as the middle term, as kings on the edge of the board will still be unable to attack each other. It is also strictly greater than the middle term, because now we add the configurations where kings are placed in the middle gap between boards. □

A direct effect of these inequalities are the following more useful formulas:

$$F(m_1, m_2, \ldots, m_{d-1}, m_d, N)^{\frac{n_1}{m_1} \frac{n_2}{m_2} \cdots \frac{n_d}{m_d}} > F(n_1, n_2, \ldots n_d, N)$$

and

$$F(m_1, m_2, \ldots, m_{d-1}, m_d, N)^{\frac{n_1}{m_1+1} \frac{n_2}{m_2+1} \cdots \frac{n_d}{m_d+1}} < F(n_1, n_2, \ldots, n_{d-1}, n_d, N)$$

if m_i divides n_i for all i.

The idea behind these calculations is to iteratively use the lemma. Where the lemma only modifies the value of one of the dimensions, here the term for all dimensions except one have been modified by repeated use of the lemma, targeting a new dimension each time. This can be done, because at each step all other terms are left constant, and due to the symmetry of the board, the relationship holds regardless of which term we are modifying.

Theorem 5. *Let λ_D be the largest eigenvalue for a transition matrix corresponding to boards of D dimensions, where $D = \{n_1, n_2, \ldots, n_d\}$, and these boards are being stacked together along the $(d+1)$ dimension.*

$$(\lambda_D)^{\frac{1}{\prod_i (n_i+1)}} < \eta_{(d+1)} < (\lambda_D)^{\frac{1}{\prod_i n_i}}$$

Proof. Using Lemma 4, solving for the bounds becomes a matter elementary calculus.

$$\begin{aligned} & \lambda_D^{\frac{1}{n_1 n_2 \cdots n_d}} \\ = \; & \left(\lim_{N\to\infty} F(D,N)^{\frac{1}{N}}\right)^{\frac{1}{n_1 n_2 \cdots n_d}} \\ = \; & \lim_{\forall m, N\to\infty} F(D,N)^{\frac{1}{N} \frac{m_1}{n_1} \frac{m_2}{n_2} \cdots \frac{m_d}{n_d} \frac{1}{m_1 m_2 \ldots m_d}} \\ > \; & \lim_{\forall m, N\to\infty} F(m_1, m_2, \ldots, m_d, N)^{\frac{1}{N} \frac{1}{m_1 m_2 \cdots m_d}} \\ = \; & \eta_{d+1} \end{aligned}$$

and

$$\begin{aligned} & \lambda_D^{\frac{1}{(n_1+1)(n_2+1)\ldots(n_d+1)}} \\ = \; & \left(\lim_{N\to\infty} F(D,N)^{\frac{1}{N}}\right)^{\frac{1}{(n_1+1)(n_2+1)\ldots(n_d+1)}} \\ = \; & \lim_{\forall m, N\to\infty} F(D,N)^{\frac{1}{N} \frac{m_1}{n_1+1} \frac{m_2}{n_2+1} \cdots \frac{m_d}{n_d+1} \frac{1}{m_1 m_2 \ldots m_d}} \\ < \; & \lim_{\forall m, N\to\infty} F(m_1, m_2, \ldots, m_d, N)^{\frac{1}{N} \frac{1}{m_1 m_2 \cdots m_d}} \\ = \; & \eta_{d+1} \end{aligned}$$

□

Using calculated eigenvalues, we have determined the following bounds.

$$
\begin{array}{llllll}
\lambda_{(2,25)} & = & 504,741.03754 & (\lambda_{(2,25)})^{1/78} & = & 1.183358 \\
 & & & (\lambda_{(2,25)})^{1/50} & = & 1.300353 \\
\lambda_{(2,2,25)} & = & 501,678,518.8 & (\lambda_{(2,2,25)})^{1/234} & = & 1.08938 \\
 & & & (\lambda_{(2,2,25)})^{1/100} & = & 1.221812 \\
\lambda_{(5,6)} & = & 401.40192924 & (\lambda_{(5,6)})^{1/42} & = & 1.153427 \\
 & & & (\lambda_{(5,6)})^{1/30} & = & 1.221198 \\
\lambda_{(3,3,4)} & = & 248.85506094 & (\lambda_{(3,3,4)})^{1/80} & = & 1.07139 \\
 & & & (\lambda_{(3,3,4)})^{1/36} & = & 1.16561
\end{array}
$$

Thus:

$$
\begin{array}{ccccc}
1.1833 & < & \eta_3 & < & 1.2212 \\
1.0894 & < & \eta_4 & < & 1.1656
\end{array}.
$$

5 η, The Sequence

The different values of η can be combined to form an infinite sequence where the d^{th} term in the sequence is η_d.

Theorem 6. $2^{\frac{1}{2^d}} < \eta_d < (1+2^d)^{2^{\frac{1}{2^d}}}$.

Proof. Consider a board of size $D = \{n_1, n_2, \ldots, n_d\}$, where $n_i = 2k \forall i$ for arbitrary value $k \in \mathbb{Z}^+$. Divide the board into k^d hypercube blocks, each block of width 2.

It is clear that each block can contain at most 1 king. Thus, there are $1 + 2^d$ possible configurations per block. It is clear that the upper bound is an over-estimate because it allows for interference between kings in different blocks.

Another estimate for the number of configurations possible is to only allow a king in the uppermost easterly square. With this situation, we get $F(1) = 2$ configurations per block. This figure discounts any interference between kings in separate blocks, but underestimates the total number of configurations. It is thus a valid lower bound.

$$(2)^{k^d} < F(D) < (1+2^d)^{k^d}$$

$$\lim_{k\to\infty}(2)^{\frac{k^d}{(2k)^d}} < \eta_d < \lim_{k\to\infty}(1+2^d)^{\frac{k^d}{(2k)^d}}$$

$$2^{\frac{1}{2^d}} < \eta_d < (1+2^d)^{2^{\frac{1}{2^d}}}$$

□

As we can see, the sequence of η's diminishes very rapidly. The increased accuracy of the bounds further into the sequence is misleading. These bounds grow farther apart as the number of dimensions grow, but more significant digits seem to appear. This is because the leading unit digit must be taken as "significant." If the *log* of these values were to be used, only a single significant digit would be known (if that).

η_1	η_2	η_3	η_4	η_{10}	$\dots$	η_{20}
≈ 1.62	≈ 1.34	≈ 1.2	≈ 1.1	≈ 1.003	$\dots$	≈ 1.000

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