DIVISIBILITY OF AN EIGENFORM BY ANOTHER EIGENFORM

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ABSTRACT. We prove a consequence of Maeda's conjecture that an eigenform is a factor of a cuspidal eigenform only when it is forced to be for dimension consideration. A similar result for Eisenstein series is also shown. We then relate the factorization of eigenforms to linear independence of Rankin-Selberg L-values.

1. PRELIMINARIES

We would like to consider some problems on factoring eigenforms in full level, $\Gamma = SL_2(\mathbb{Z})$. It is known that there is a basis of eigenforms for a modular space or modular cusp space, and that by basic linear algebra they are orthogonal under the Petersson inner product, as their eigenvalues are different. In this paper we shall consider some questions and cases for the situation $h = fg$ where h is a normalized eigenform. Several similar problems have been considered in terms of products of eigenforms. In particular products of Eisenstein series were invesigated in [11]. Independently [2] and [6] show that the product of two eigenforms is an eigenform finitely many times. More generally [3] shows that the product of any number of eigenforms is only an eigenform finitely many times. This paper further generalizes these results.

Throughout this paper we will use $\{g_1, ..., g_b\}$ as an eigenform basis for $S_{wt(g)}$, the space of cuspforms of weight $wt(g)$. Similarly let $\{h_1, ..., h_d\}$ be a normalized eigenform basis for the cuspidal space of weight $wt(h)$. Because h can be assumed to be fixed, we will (sometimes) also denote the Hecke operator $T_{n,wt(h)}$ on $S_{wt(h)}$ by T_n. We use the following definition for an Eisenstein series.

Definition 1.1. *The weight k Eisenstein series is the modular form given by*

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

See [10] for more detail on $E_k(z)$. In this paper we will investigate the possibility of factoring the eigenform h as $h = fg$ with one of f, g an eigenform. There are three possible cases with h and f eigenforms:

(1) h and f are cuspidal eigenforms
(2) h is a cuspidal eigenform, f is a noncuspidal eigenform, i.e. f is an Eisenstein series.
(3) h and f are noncuspidal eigenforms, i.e. both f and g are Eisenstein series.

Toward this end we will first give some general lemmas.

Definition 1.2. *Let $\mathbb{F} \subseteq \mathbb{C}$ be a subfield. A subspace of $S \subseteq S_m$ is said to be $\mathbb{F}$-rational if it is stable under the Galois action: $\sigma(S) = S$ for all $\sigma \in Gal(\mathbb{C}/\mathbb{F})$. Here an automorphism σ acts on modular forms through their Fourier coefficients.*

The following is our $\mathbb{F}$-rational subspace lemma.

Lemma 1.3. *If a proper $\mathbb{F}$-rational subspace S of S_m contains an eigenform, then $T_{n,m}(x)$, the characteristic polynomial of $T_{n,m}$, is reducible over $\mathbb{F}$ for all n.*

Proof. Let $S \subset S_m$ be a proper $\mathbb{F}$-rational subspace containing an eigenform $h \in S$. Then define

$$R := \langle \sigma(h) | \sigma \in Gal(\mathbb{C}/\mathbb{F}) \rangle_{\mathbb{C}} \leq S$$

which is an $\mathbb{F}$-rational subspace.

Then $S_m = R \oplus R^{\perp}$, both of which are stable under Hecke operators because they have eigenform bases. Denote $T_{n,m}(x)$ as the characteristic polynomial of $T_{n,m}$ and $T_{n,m}|_R(x)$ as the restriction of $T_{n,m}(x)$ to R.

Thus $T_{n,m}(x) = T_{n,m}|_R(x) \cdot T_{n,m}|_{R^{\perp}}(x)$.

The roots of $T_{n,m}|_R(x)$ are permuted by $\sigma \in Gal(\mathbb{C}/\mathbb{F})$ because $\sigma(T_{n,m}|_R(x)) = T_{n,m}|_R(x)$; thus $T_{n,m}|_R(x)$ has coefficients in $\mathbb{F}$.

Hence $T_{n,m}|_{R^{\perp}}(x)$ also has $\mathbb{F}$-rational coefficients, and so we have that $T_{n,m}(x)$ is reducible over $\mathbb{F}$. □

Corollary 1.4. *If for some n, $T_{n,m}(x)$ is irreducible over $\mathbb{F}$, then no proper $\mathbb{F}$-rational subspace of S_m can contain an eigenform.*

We give a remark on the irreducibility assumption of $T_{n,m}(x)$ over $\mathbb{F}$. If $[\mathbb{F} : \mathbb{Q}] \geq \dim(S_m)$, then it could be the case that $T_{n,m}(x)$ is reducible over $\mathbb{F}$, in particular if $\mathbb{F}$ is the splitting field of $T_n(x)$. This will not cause us great chagrin because we will only be interested in fields of small degree. In this case Maeda's conjecture implies the irreducibility of all $T_{n,m}(x)$. See section 6.

2. f is a cuspidal eigenform

In this section we will see when there is a factorization $h = fg$ with both h and f cuspidal eigenforms. We will need to separate out the following cases for the theorem. We find by dimension computation [10] that $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$ with f cuspidal only as in the following lemma.

Lemma 2.1. *One has* $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$ *in and only in the following cases.*

$$\begin{aligned} wt(f) = 12, wt(g) &\equiv 4, 6, 8, 10, 12, 14 \mod (12) \\ wt(f) = 16, wt(g) &\equiv 4, 6, 10, 12 \mod (12) \\ wt(f) = 18, wt(g) &\equiv 4, 8, 12 \mod (12) \\ wt(f) = 20, wt(g) &\equiv 6, 12 \mod (12) \\ wt(f) = 22, wt(g) &\equiv 4, 12 \mod (12) \\ wt(f) = 26, wt(g) &\equiv 12 \mod (12) \end{aligned}$$

The case that $\dim(S_{wt(f)}) = \dim(S_{wt(h)})$ is subsumed in the above because this is the case where $\dim(M_{wt(g)}) = 1$, and so $h = fE_{wt(h)-wt(f)}$ is a factorization into eigenforms which occurs only finitely many times, in particular when $\dim(S_{wt(h)}) = \dim(M_{wt(g)}) = 1$. Here M_n denotes the space of modular forms of weight n. In all these cases it turns out that f is from a dimension one space.

We will use the above lemma and the following definition in the first theorem.

Definition 2.2. *Given a normalized eigenform* f*, let* $\mathbb{F}_f$ *denote the field containing its fourier coefficients. In particular, if* $f = \sum a_n q^n$ *then* $\mathbb{F}_f = \mathbb{Q}(a_0, a_1, a_2, ...)$.

Note that $\mathbb{F}_f/\mathbb{Q}$ is a finite extension and has $\dim(\mathbb{F}_f) \leq \dim(S_{wt(f)})$. See [12] for more information on these spaces.

Theorem 2.3. *If for some* n*,* $T_n(x)$ *is irreducible over every field* $\mathbb{F}$ *of degree less than* $d = \dim(S_{wt(h)})$*, then a cuspidal eigenform* $h \in S_{wt(h)}$ *can be factored as* $h = fg$ *with* f *a cuspidal eigenform and* g *a modular form if and only if* $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$*. These are precisely the cases in lemma 2.1.*

Proof. Suppose that we are not in the cases given in lemma 2.1. That is $\dim(S_{wt(h)}) > \dim(M_{wt(g)})$. Consider the space $fM_{wt(g)} = \langle fE_{wt(g)}, fg_1, fg_2, ..., fg_u \rangle$. By construction this is a $\mathbb{F}_f$-rational subspace of $S_{wt(h)}$ of dimension $\dim(M_{wt(g)})$. If $\dim(S_{wt(h)}) > \dim(M_{wt(g)})$, it is a proper $\mathbb{F}_f$-rational subspace of $S_{wt(h)}$. On the other hand, this space contains an eigenform fg. Hence by lemma 1.3 we know that $T_n(x)$ is reducible over $\mathbb{F}_f$ for all n. This contradicts the premises because $[\mathbb{F}_f : \mathbb{Q}] < \dim(S_{wt(f)}) < d$, and we have just factored $T_n(x)$ over $\mathbb{F}_f$.

In the other case, where $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$, by comparing the weights f must be of weight $4, 6, ..., 22, 26$. Because f is cuspidal we must have $\dim(S_{wt(f)}) \geq 1$ and so in fact $\dim(S_{wt(f)}) = 1$. In these cases we can use linear algebra to construct a factorization $h = fg$. In theory one can notice that $fM_{wt(g)} \subset S_{wt(h)}$ and is of the same dimension, and so such a construction must exist. In practice one may write g as a linear combination of a triangular basis meaning that the nth basis element has the first $n-1$ Fourier coefficients equal to zero. This allows one to use the coefficient of the nth basis element to determine the nth Fourier coefficient in the product fg. Because the dimensions are equal, there are just enough basis elements to construct a product fg equal to h. □

Note that if $\dim(S_{wt(h)}) = 1$, then the above reduces into the cases that are treated in [6] and [2].

Corollary 2.4. *If for some n, $T_n(x)$ is irreducible over every field $\mathbb{F}$ of degree less than d and $h = fg$ with h and f cuspidal eigenforms, then f comes from one a dimensional space, i.e. $wt(f) = 12, 16, 18, 20, 22, 26$.*

In theorem 2.3 we have assumed that $T_n(x)$ is irreducible over every field of degree less than d. If $[\mathbb{F}_f : \mathbb{Q}] = \dim(S_{wt(h)})$ then $T_n(x)$ could potentially be reducible over $\mathbb{F}_f$. However, from the fact that $[\mathbb{F}_f : \mathbb{Q}] \leq \dim(S_{wt(f)})$ this does not happen often, as we would need $\dim(S_{wt(f)}) = \dim(S_{wt(h)})$ which happens only finitely many times (in fact 17 times) because this forces $\dim(M_{wt(g)}) = 1$ and so all three of f, g, h are eigenforms. Otherwise, we have $[\mathbb{F}_f : \mathbb{Q}] < \dim(S_{wt(h)})$ in which case we have reason to believe that T_n is irreducible from Maeda's conjecture, see section 6.

3. f is an Eisenstein series

In this section we will see when there is a factorization $h = fg$ with h a cuspidal eigenform, and f a non-cuspidal eigenform, i.e. an Eisenstein series. We will need to separate out the following cases for the theorem. Using the dimension formula [10] we get the following.

Lemma 3.1. *One has $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$ in and only in the following cases:*

$$\begin{aligned} wt(f) = 4, wt(g) &\equiv 0, 4, 6, 10 \mod (12) \\ wt(f) = 6, wt(g) &\equiv 0, 4, 8 \mod (12) \\ wt(f) = 8, wt(g) &\equiv 0, 6 \mod (12) \\ wt(f) = 10, wt(g) &\equiv 0, 4 \mod (12) \\ wt(f) = 14, wt(g) &\equiv 0 \mod (12) \end{aligned}$$

Theorem 3.2. *If for some n, $T_n(x)$ is irreducible over $\mathbb{Q}$, then a cuspidal eigenform $h \in S_{wt(h)}$ can be factored as $h = fg$ where f is an Eisenstein series, g is a modular form if and only if $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$. These are precisely the cases in lemma 3.1.*

Proof. Suppose that we are not in the cases given in lemma 3.1, i.e. $\dim(S_{wh(t)}) > dim(S_{wt(g)})$. Let $fS_{wt(g)} = \langle fg_1, fg_2, ..., fg_u \rangle$. As f has rational coefficients, this is a $\mathbb{Q}$-rational subspace of $S_{wt(h)}$. As $\dim(S_{wt(h)}) > \dim(S_{wt(g)})$, in a similar manner as the proof of theorem 2.3 we find that $T_n(x)$ is reducible over $\mathbb{Q}$ for all n.

In the cases of lemma 3.1 where $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$ we can use linear algebra to construct a factorization $h = fg$: write g as a linear combination of basis elements, and solve for the coefficients using the Fourier expansion of h, similar to the argument used in the proof of theorem 2.3. □

If $\dim(S_{wt(h)}) = 1$, then the above reduces to the cases that are given in [2] and [6].

Corollary 3.3. *Suppose for some n, $T_n(x) = T_{n,wt(h)}(x)$ is irreducible over $\mathbb{Q}$. Then, E_s divides the cuspidal eigenform h if and only if $\dim(M_s) = 1$, i.e. $s = 4, 6, 8, 10, 14$.*

4. h is an Eisenstein series

In this section we will see when there is a factorization $h = fg$ with h and f noncuspidal eigenforms, that is, Eisenstein series. In particular we are investigating $E_k = E_s g$; we will need to separate out the following cases for the theorem. Denote $l = wt(g) = k - s$. Direct dimension computation gives:

Lemma 4.1. *One has* $\dim(M_k) = \dim(M_l)$ *in and only in the following cases:*

$$\begin{aligned} s = 4, l \equiv 0, 4, 6, 10 \mod (12) \\ s = 6, l \equiv 0, 4, 8 \mod (12) \\ s = 8, l \equiv 0, 6 \mod (12) \\ s = 10, l \equiv 0, 4 \mod (12) \\ s = 14, l \equiv 0 \mod (12) \end{aligned}$$

We will also need to define an auxiliary polynomial φ_r.

Definition 4.2. *Let* $\varphi_r = \prod(x - j_i)$*, where the product runs over all the* j*-zeros of* E_r *except for 0 and 1728. (Under the* j*-mapping* i *and* ρ *correspond to* 0 *and* 1728 *respectively).*

Note that φ_r is monic with rational coefficients. See [5] for more information on this function.

In this case our main result can now be presented.

Theorem 4.3. *If* φ_k *is irreducible over* $\mathbb{Q}$*, then* E_s *divides* E_k *when and only when* $\dim(M_k) = \dim(M_l)$*. These are precisely the cases in lemma 4.1.*

Proof. If $E_k = E_s g$, then φ_s divides φ_k. Because φ_k is irreducible, this means that either φ_s is a constant, or φ_s is a constant multiple of φ_k. Because the degree of φ_r equals $\dim(S_r)$ [5], this means that either $\dim(M_s) = 1$ or $\dim(M_s) = \dim(M_k)$.

If $\dim(M_s) = \dim(M_k)$, then $\dim(M_l) = 1$, so that we have a factorization of an Eisenstein series as two Eisenstein series. By [2] and [6] this can occur only finitely many times, in which case $\dim(M_s) = 1$. Hence if $E_k = E_s g$, then $\dim(M_s) = 1$.

Now suppose $\dim(M_s) = 1$. We then see that $\dim(M_k) = \dim(M_l)$ because each are one more than the degree of E_k and g respectively. Altogether if $E_k = E_s g$ then $\dim(M_k) = \dim(M_l)$.

Conversely if $\dim(M_k) = \dim(M_l)$ then we can use linear algebra to construct a noncuspidal g such that $E_k = E_s g$ by writing g in terms of basis elements of $\dim(M_l)$ and using the Fourier expansion of h, similar to the argument used in the proof of theorem 2.3. □

We have reason to believe that φ_k is irreducible, as commented upon in [5] and verified for weights less than 700. Later we verified that φ_r is irreducible for weights up to 1500. See section 6.

5. Relationship to L-values

In this section we investigate the relationship between the above factorizations and Rankin Selberg L-values. Here we consider the second case that $f = E_s$. In particular, we are investigating $h = E_s g$. As before we let $\{h_1, ..., h_d\}$ be a normalized eigenform basis for the cuspidal space of weight $wt(h)$.

We will translate the result of theorem 3.2 into an equivalent theorem regarding Rankin-Selberg L-values. We know that $\langle h_j, h_i \rangle = 0$ for $i = 1, 2, ..., j-1, j+1, ..., d$ which gives us $d-1$ orthogonality constraints. Suppose we have

$$c_1 \langle E_s g_1, h_i \rangle + \cdots + c_b \langle E_s g_b, h_i \rangle = \langle h_1, h_i \rangle = 0.$$

The Rankin-Selberg method of inner products can be expressed as, with $k = wt(g)$,

$$\langle f, E_s g \rangle = (4\pi)^{-s+k-1} \Gamma(s+k-1) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^{s+k-1}} = (4\pi)^{-s+k-1} \Gamma(s+k-1) L(f \times g, s)$$

See [1] for more information. In our case, if we divide by the factor $(4\pi)^{-wt(h)-1}(wt(h)-1)!$ we obtain

$$c_1 L(g_1, h_i) + \cdots + c_b L(g_b, h_i) = 0$$

where for normalized cuspidal eigenforms $g = \sum_{i=1}^{\infty} a_i q^i$, $h = \sum_{i=1}^{\infty} b_i q^i$ we used the notation

$$L(g, h) := L(g \times h, wt(h) - 1) = \sum_{i=1}^{\infty} \frac{a_i \overline{b_i}}{i^{wt(h)-1}}.$$

In particular we have another set of linear equations. Express the coefficients in vector form:

$$\left\{ \begin{bmatrix} L(g_1, h_1) \\ \vdots \\ L(g_1, h_{j-1}) \\ L(g_1, h_{j+1}) \\ \vdots \\ L(g_1, h_d) \end{bmatrix}, ..., \begin{bmatrix} L(g_b, h_1) \\ \vdots \\ L(g_b, h_{j-1}) \\ L(g_b, h_{j+1}) \\ \vdots \\ L(g_b, h_d) \end{bmatrix} \right\} \tag{5.1}$$

Proposition 5.2. *Assume that for some n, $T_n(x)$ is irreducible over every field $\mathbb{F} \subseteq \mathbb{C}$ of degree less than d. Then for $d > b$, the vectors of L values given in (5.1) are linearly independent over $\mathbb{C}$, and when for $d = b$ there is a single dependency relation.*

An easier form to deal with these L-values is in the following matrix.

$$\begin{bmatrix} L(g_1, h_1) & \cdots & L(g_b, h_1) \\ \vdots & \ddots & \vdots \\ L(g_1, h_{j-1}) & \cdots & L(g_b, h_{j-1}) \\ L(g_1, h_{j+1}) & \cdots & L(g_b, h_{j+1}) \\ \vdots & \ddots & \vdots \\ L(g_1, h_d) & \cdots & L(g_b, h_d) \end{bmatrix} \tag{5.3}$$

We will want to use an easier version, when $i = 1$:

$$\begin{bmatrix} L(g_1, h_2) & \cdots & L(g_b, h_2) \\ \vdots & \ddots & \vdots \\ L(g_1, h_d) & \cdots & L(g_b, h_d) \end{bmatrix} \tag{5.4}$$

Proposition 5.5. *Assume that for some n, $T_n(x)$ is irreducible over all subfields $\mathbb{F} \subseteq \mathbb{C}$ of degree less than d. Then the matrix given in (5.3) is of full rank.*

Proof. First prop 5.2 and prop 5.5 are via basic linear algebra merely different ways of stating the same claim.

Assume that for some n, $T_n(x)$ is irreducible over all fields of degree less than d. So we can apply theorem 3.2. There are two cases.

Case $d > b$: Suppose there is a solution $[c_1, ..., c_b]^T$ to the matrix equation $L\overrightarrow{x} = \overrightarrow{0}$. We must show that $[c_1, ..., c_b]^T = \overrightarrow{0}$. We have:

$$c_1 L(g_1, h_i) + \cdots + c_b L(g_b, h_i) = 0.$$

By using the Rankin-Selberg method and denoting $g := c_1 E_s g_1 + \cdots c_b E_s g_b$ we have $\langle g, h_i \rangle = 0$ for $i = 2, ..., d$. Hence g is orthogonal to each of $h_2, h_3, ..., h_d$. Hence by linear algebra $E_s g = c h_1$. By the assumption, this cannot occur nontrivially, and so $g = 0$ and $c = 0$. In particular, $c_1 = \cdots = c_b = 0$.

Case $d = b$: Because L is underdetermined there clearly are nonzero solutions. We must show that there is only one linearly independent solution. Suppose there are two solutions $[c_1, ..., c_b]^T$ and $[c'_1, ..., c'_b]^T$ to the matrix equation $L\overrightarrow{x} = \overrightarrow{0}$. Similar to before we construct $g := c_1 E_s g_1 + \cdots c_b E_s g_b$ and $g' := c'_1 E_s g_1 + \cdots c'_b E_s g_b$ which satisfy, respectively, $E_s g = c h_1$, $E_s g' = c' h_1$ for some $c, c' \in \mathbb{C}$. By assumption g and g' are scalar multiples of each other. Hence c and c' are linearly dependent.

□

6. Conclusions and Maeda's Conjecture

On the positive side of factorization of cuspidal eigenforms, we have the following.

Proposition 6.1. *There are infinitely many examples of cuspidal non-eigenforms g such that $E_s g$ is a cuspidal eigenform.*

Proof. From lemma 3.1 we see that there are twelve infinite classes such as $E_4 g$ with $wt(g) = 24$ such that the matrix (5.3) is underdetermined. That is, the dimension of the solution space is 1, so that we see that there is a cuspform g such that $E_s g$ is an eigenform. From the proof of prop 5.5 we see that g is not an eigenform for $b > 1$ because at least two of the c_i must be nonzero. □

Now that we know there are many equations of the form $E_s g = h$ with h an eigenform, we present an example of one such case.

Example 6.2. *As an example of a nontrivial factorization, consider that for both weight 28 cusp eigenforms h_1, h_2, there are weight 24 cusp non-eigenform f_1, f_2 so that $E_4 f_1 = h_1$ and $E_4 f_2 = h_2$. (Note also that this is the smallest such example - in all smaller cases it turns out that f_i is an eigenform - which are actually the cases presented in* [2]*, and* [6]*.)Working this out, we find that*

$$E_4(aE_{12}\Delta + b\Delta^2) = cE_{16}\Delta + dE_4\Delta^2$$

where the right hand side above is an eigenform, and the parenthesized factor on the left is not. These numbers are given in the table below

a	*1*
b	$-\frac{3075516}{691} - 108\sqrt{18209}$
c	*1*
d	$-\frac{14903892}{3617} - 108\sqrt{18209}$

So we know that as long as some Hecke operator T_n is irreducible over all fields $\mathbb{F}$ of degree less than d, then a cuspidal eigenform can be factored only in the cases in lemma 2.1 and lemma 3.1. We have reason to believe that this is a sound assumption. In particular, we have the following conjecture due to Maeda.

Conjecture 6.3 (Maeda)**.** *The Hecke algebra over $\mathbb{Q}$ of $S_m(SL_2(\mathbb{Z}))$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group $\mathcal{S}_n$ (with $n = dim S_m(SL_2(\mathbb{Z}))$).*

This conjecture appeared in [7], and in the same paper was verified for weights less than 469. Later it was verified up to weight 2000 [4] and weight 3000 [9].

By the following proposition Maeda's conjecture implies that $T_{n,m}(x)$ is irreducible over all fields $\mathbb{F}$ with $[\mathbb{F} : \mathbb{Q}] < \dim(S_m)$, which was used as an assumption in theorem 2.3.

Proposition 6.4. *Let $P(x) \in \mathbb{Q}[x]$ be a degree d polynomial. Let K_P be its splitting field. Assume $[K_P : \mathbb{Q}] = d!$, i.e., $Gal(K_P/\mathbb{Q}) \cong S_d$. If P factors over K, then $[K : \mathbb{Q}] \geq d$.*

Proof. Suppose P is reducible over K and $[K : \mathbb{Q}] < d$. Write $P = QR$, where $Q, R \in K[x]$ are polynomials of degrees d_1, d_2 and have splitting fields K_Q, K_R respectively. Then $d_1 + d_2 = d$.

Write $m := [K_Q : K], n := [K_R : K]$. Then

$$d_1!d_2! \geq mn \geq [K_Q K_R : K] \geq [K_P : K] > (d-1)!,$$

which occurs if and only if $d_1 = 0$ or $d_2 = 0$. Hence one of Q or R is a constant, so that P is irreducible over K.

□

Let us now synthesize theorems 2.3 and 3.2 with Maeda's conjecture.

Theorem 6.5. *Maeda's conjecture implies that an eigenform is a factor of another cuspidal eigenform precisely in cases listed in lemma 2.1 and lemma 3.1.*

As a final note, consider that Maeda's conjecture has been verified in [7] for weights less than 469, in [4] for weights up to 2000 and in [9] for weights up to 3000. The irreducibility of φ_r has been verified in [5] up to weight 700, and by our own computations up to weight 1500. This means that our theorems are unconditionally true for these weights.

As a final remark on our computations concerning φ_r, we used an equation presented in [8] which gives an equality:

$$\frac{E_r}{E_4^a E_6^b \Delta^c} = \varphi_r(j(\tau)),$$

where $4a + 6b + 12c = r$, with $0 \leq a \leq 2$, $0 \leq b \leq 1$. We then computed f modulo small primes p and in each case found a prime p such that φ_r is irreducible. There is no reason other than runtime that the highest weight computed was 1500.

References

[1] Daniel Bump. The Rankin-Selberg method: a survey. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 49–109. Academic Press, Boston, MA, 1989.

[2] W. Duke. When is the product of two Hecke eigenforms an eigenform? In *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, pages 737–741. de Gruyter, Berlin, 1999.

[3] Brad A. Emmons and Dominic Lanphier. Products of an arbitrary number of Hecke eigenforms. *Acta Arith.*, 130(4):311–319, 2007.

[4] D. W. Farmer and K. James. The irreducibility of some level 1 Hecke polynomials. *Math. Comp.*, 71(239):1263–1270 (electronic), 2002.

[5] Ernst-Ulrich Gekeler. Some observations on the arithmetic of Eisenstein series for the modular group $\mathrm{SL}(2, \mathbb{Z})$. *Arch. Math. (Basel)*, 77(1):5–21, 2001. Festschrift: Erich Lamprecht.

[6] Eknath Ghate. On monomial relations between Eisenstein series. *J. Ramanujan Math. Soc.*, 15(2):71–79, 2000.

[7] Haruzo Hida and Yoshitaka Maeda. Non-abelian base change for totally real fields. *Pacific J. Math.*, (Special Issue):189–217, 1997. Olga Taussky-Todd: in memoriam.

[8] M. Kaneko and D. Zagier. Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 97–126. Amer. Math. Soc., Providence, RI, 1998.

[9] Seth Kleinerman. Some computations in support of maeda's conjecture. http://modular.math.washington.edu/edu/Fall2003/252/final_project/seth/252paper.pdf, 2010.

[10] Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[11] R. A. Rankin. Elementary proofs of relations between Eisenstein series. *Proc. Roy. Soc. Edinburgh Sect. A*, 76(2):107–117, 1976/77.

[12] Goro Shimura. *Elementary Dirichlet series and modular forms*. Springer Monographs in Mathematics. Springer, New York, 2007.

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