On unique realizations of domination chain parameters

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Abstract

The domination chain $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$, which holds for any graph G, is the subject of much research. In this paper, we consider the maximum number of edges in a graph having one of these domination chain parameters equal to 2 through a unique realization. We show that a specialization of the domination chain still holds in this setting.

Keywords: vertex domination, vertex independence, vertex irredundance

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1 Introduction

Domination is a well-studied field in graph theory. In particular, the domination chain

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$$

which holds for every graph G, has itself been studied in many papers. In our work, we consider a specialization of this chain as it pertains to unique realizations of these parameters.

Unique minimum dominating sets were first discussed in the 1994 paper by Gunther, Hartnell, Markus, and Rall [8] where trees were the primary class of graphs studied. Since then, unique minimum dominating sets have been studied in block graphs [2], cactus graphs [4], and Cartesian products [9] and [10]. In [3] and [7] the problem of determining the maximum number of edges in a graph having a unique γ-set was considered.

Unique realizations of the other domination chain parameters have also been considered. For example, unique ir-sets and unique i-sets were considered in [6], unique β_0-sets were considered in [11] and [12], while unique β_0-sets, Γ-sets, and IR-sets were considered in [5].

In our work to follow, we build on the work of [3] and [7] and consider the maximum number of edges in a graph having a unique ir, i, β_0, Γ, or IR-set of cardinality 2. In so doing, we show that the domination chain above still holds in this different setting.

2 Definitions

Throughout our work, G denotes a finite, simple, undirected graph, $V(G)$ denotes the vertex set of G, and $E(G)$ denotes the edge set of G. For $v \in V(G)$, the *open neighborhood of* v, denoted by $N(v)$, is defined by $N(v) = \{u : uv \in E(G)\}$, while the *closed neighborhood of* v, denoted $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. In regards to dominating sets discussed below, we say that a vertex *dominates* all vertices in its closed neighborhood. Additionally, for $S \subseteq V(G)$, the *open neighborhood of* S, denoted $N(S)$, is defined by $N(S) = \cup_{v \in S} N(v)$, while *the closed neighborhood of* S, denoted by $N[S]$, is defined by $N[S] = N(S) \cup S$. Given a set $S \subseteq V(G)$ and a vertex $x \in S$, an *external private neighbor of* x *with respect to* S is a vertex $v \in V(G) - S$ such that $N[v] \cap S = \{x\}$. x itself is a *self-private neighbor with respect to* S if $N[x] \cap S = \{x\}$. Finally, $G[S]$ denotes the subgraph of G induced by S.

Let $S \subseteq V(G)$.

- S is *irredundant* if for all $v \in S$, $N[v] - N[S - \{v\}] \neq \emptyset$. Equivalently, S is irredundant if for all $v \in S$, v either has an external private neighbor or is a self-private neighbor.
- S is *dominating* if $N[S] = V(G)$.
- S is *independent* if no two vertices in S share an edge.

The minimum cardinality of a maximal (with respect to set inclusion) irredundant set is denoted $ir(G)$, while the maximum cardinality of an irredundant set is denoted $IR(G)$. We let $\gamma(G)$ denote the minimum cardinality of a dominating set, and $\Gamma(G)$ denote the maximum cardinality of a minimal (with respect to set inclusion) dominating set. Finally, $i(G)$ denotes the minimum cardinality of a maximal (with respect to set inclusion) independent set, while $\beta_0(G)$ denotes the maximum cardinality of an independent set. These parameters are related by the following well-known result of Cockayne, Hedetniemi, and Miller.

Theorem. *[1] For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

For convenience, a maximal irredundant set of cardinality $ir(G)$ or $IR(G)$ is called an ir-set or IR-set respectively. Similarly, a minimal dominating set of cardinality $\gamma(G)$ or $\Gamma(G)$ is referred to as γ-set or Γ-set respectively, while a maximal independent set of cardinality $i(G)$ or $\beta_0(G)$ is referred to as i-set or β_0-set respectively.

Let $\mathcal{P}$ be one of ir, γ, i, β_0, Γ, or IR. In general, a graph may have many $\mathcal{P}$-sets. We are interested in graphs having a unique $\mathcal{P}$-set. We make the following definitions.

Definition 1. *Let $m^*_{\mathcal{P}}(n,k)$ denote the maximum number of edges in a graph on n vertices having a unique $\mathcal{P}$-set of cardinality k.*

Definition 2. *Let $m_{\mathcal{P}}(n,k)$ denote the maximum number of edges in an isolate free graph on n vertices having a unique $\mathcal{P}$-set of cardinality k.*

Observe that $m_{\mathcal{P}}(n,k) \leq m^*_{\mathcal{P}}(n,k)$.

With this notation now defined, our main result is as follows.

Theorem 1. *For $n \geq 6$*

$$m_{ir}(n,2) = m_{\gamma}(n,2) \leq m_i(n,2) \leq m_{\beta_0}(n,2) = m_{\Gamma}(n,2) = m_{IR}(n,2)$$

and

$$m^*_{ir}(n,2) = m^*_{\gamma}(n,2) \leq m^*_i(n,2) \leq m^*_{\beta_0}(n,2) = m^*_{\Gamma}(n,2) = m^*_{IR}(n,2).$$

We prove this theorem by computing, or recalling in the case of $m_{\gamma}(n,2)$, the exact values for each of the parameters.

In Section 3, we collect the results from [3] and [7] which we need for our discussion of unique irredundant sets in Section 4. In Section 5, we turn our attention to unique minimum independent dominating sets of cardinality two, while in Section 6 we consider unique β_0-, Γ-, and IR-sets of cardinality k for $k \geq 2$. Finally, in Section 7, we pose several open problems.

3 Unique minimum dominating sets

In this section, we briefly collect the results from [3] and [7] that we will need in our work to come. We first note the following result from [3].

Proposition 1. *[3] For $n \geq 3$, $m_{\gamma}(n,1) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$.*

For comparison purposes in Section 5 to follow, we also note the following result.

Theorem 2. *[3] For $k \geq 2$, $m_{\gamma}(3k,k) = 2k + 2\binom{k}{2}$.*

The following result, from [7], will be used in Section 4 below.

Theorem 3. *[7] For $n \geq 6$, $m_{\gamma}(n,2) = \binom{n-2}{2}$.*

By combining Proposition 1 and Theorem 3, we see the following.

Proposition 2. *For $n \geq 6$, $m^*_{\gamma}(n,2) = m_{\gamma}(n-1,1) = \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$.*

Proof. Suppose G is a graph on $n \geq 6$ vertices having a unique γ-set of cardinality 2. Note that G has at most one isolated vertex. If G is isolate free, then $|E(G)| \leq \binom{n-2}{2}$ by Theorem 3. However, if G has one isolate, call it v, and a unique γ-set of cardinality 2, then $G - v$ necessarily induces an isolate free graph having a unique γ-set of cardinality 1. Hence, if G has an isolate, then $|E(G)| \leq \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$ by Proposition 1. Since this upper bound is strictly greater than $\binom{n-2}{2}$ and can be achieved, our result follows. $\square$

4 Unique minimum maximal irredundant sets

We now turn our attention to the maximum number of edges in a graph G having a unique ir-set of cardinality 2. Note that the only graph on three or fewer vertices having a unique ir-set of cardinality 2 is $\overline{K_2}$, the graph on two isolated vertices. Thus, we restrict our attention to graphs on at least four vertices. First, suppose that we allow G to have an isolated vertex (if G has two or more isolated vertices, then $ir(G) \geq 3$). In this case, G has a component of order $n-1$, call it C, which necessarily satisfies $ir(C) = 1$. Since $ir(C) = 1$ if and only if $\gamma(C) = 1$, we see that, in fact, C has a unique γ-set of cardinality 1. Hence, by Proposition 1, C, and thus G, has at most $\binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$ edges. This bound is achieved by the graph $K_1 \cup H$ where H is a graph on $n-1$ vertices having a maximum number of edges subject to the condition that exactly one vertex is of degree $n-2$. Thus, we see that $m^*_{ir}(n,2) \geq \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$. Thus, by Proposition 2, if $n \geq 6$, then $m^*_{ir}(n,2) \geq m^*_{\gamma}(n,2)$.

Next, we restrict ourselves to isolate free graphs. It can be readily checked that no isolate free graph on two, three, or four vertices has a unique ir-set of cardinality 2. Among the isolate free graphs on five vertices, twelve satisfy $ir(G) = 2$, and none has a unique ir-set. Thus, we restrict ourselves to graphs on $n \geq 6$ vertices.

Suppose G is an isolate free graph on $n \geq 6$ vertices having a unique ir-set of cardinality 2, call it D. D is either a dominating set, or it is not. We first consider the case when D is not a dominating set. Observe that in this case, $ir(G) < \gamma(G)$, for if $\gamma(G) = 2$ also, then D is not a unique ir-set in G.

4.1 D is not a dominating set

We begin by considering an example. Let the graph F on $n \geq 7$ vertices be constructed as follows. Let $V(F) = \{x, y, x', y', w, z, v, b_1, b_2, \ldots, b_{n-7}\}$.

Let $F[\{v, b_1, b_2, \ldots, b_{n-7}\}]$ be complete. Additionally, let

$$\begin{aligned}
N(x) &= \{y, v, x', b_1, b_2, \ldots, b_{n-7}\} \\
N(y) &= \{x, v, y', b_1, b_2, \ldots, b_{n-7}\} \\
N(v) &= \{x, y, b_1, b_2, \ldots, b_{n-7}\} \\
N(x') &= \{x, w, y', b_1, b_2, \ldots, b_{n-7}\} \\
N(y') &= \{y, z, x', b_1, b_2, \ldots, b_{n-7}\} \\
N(w) &= \{x'\} \\
N(z) &= \{y'\}.
\end{aligned}$$

The case of $n = 8$ is illustrated below for convenience.

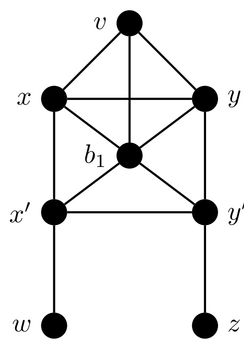

Figure 1: $n = 8$ case

We claim that $\{x, y\}$ is the unique ir-set of F.

Proof. First, note that $ir(F) \geq 2$ since $\Delta(F) < n - 1$. Consider $\{x, y\}$. This set is irredundant, since x has x' as a private neighbor and y has y' as a private neighbor. Moreover, this set is maximal irredundant since the inclusion of w, z, x', y', or $b_1, b_2, \ldots, b_{n-7}$ eliminates the private neighbor of x or y while the inclusion of v results in v not having a private neighbor. Thus, $ir(F) = 2$ and $\{x, y\}$ is an ir-set.

To see that no other two element subsets of $V(F)$ containing x are maximal irredundant, observe first that $\{x, v\}$ and $\{x, b_i\}$ for $1 \leq i \leq n - 7$ are both redundant sets. Additionally, since the sets $\{x, x', z\}, \{x, y', w\}$, and $\{x, w, z\}$ are irredundant, we see that $\{x, x'\}, \{x, y'\}, \{x, w\}$ and $\{x, z\}$ are not *maximal* irredundant.

The interested reader can verify, in a manner similar to the above, that any two element subset of $V(F)$ distinct from $\{x, y\}$ is either redundant, or is contained in a larger irredundant set. We thus have that $\{x, y\}$ is the the unique ir-set of F. $\square$

Before proceeding, we note that $|E(F)| = \binom{n-2}{2} - 2$.

We now prove the following theorem through a sequence of claims.

Theorem 4. *Let $n \geq 7$. If G is a graph of order n such that $\delta(G) \geq 1$, $ir(G) = 2$, $\gamma(G) \geq 3$, and G has a unique ir-set D, then $|E(G)| \leq \binom{n-2}{2} - 2$. Furthermore, this bound is sharp.*

Proof. Among all isolate free graphs on n vertices with domination number at least 3 and having a unique ir-set of cardinality 2, suppose that G has the maximum number of edges. Note that since $\gamma(G) \geq 3$, $\Delta(G) \leq n-3$. Let $D = \{x, y\}$ denote the unique ir-set of G. Define the following sets.

- $D_x = N(x) - N[y]$
- $D_y = N(y) - N[x]$
- $D_{xy} = N(x) \cap N(y)$
- $R = V(G) - (N[x] \cup N[y])$

We note that R is the set of vertices not dominated by x or y. Since D is not a dominating set, we have that $|R| > 0$. This implies that $xy \in E(G)$, since if $xy \notin E(G)$, then $\{x, y, w\}$, where $w \in R$, is independent and thus irredundant. The fact that $xy \in E(G)$ further implies that $D_x \neq \emptyset$ and that $D_y \neq \emptyset$.

We first consider the set R.

Claim 1. $|R| \geq 2$.

Proof of Claim: For the sake of contradiction, suppose $|R| = 1$ with $w \in R$. Since D is maximal irredundant, $\{x, y, w\}$ is redundant. Since w is a self-private neighbor with respect to $\{x, y, w\}$, we see that either w dominates D_x or w dominates D_y. Without loss of generality, assume w dominates D_x. In this case, $\{y, w\}$ is a dominating set of G, a contradiction. Hence, $|R| \geq 2$.

Claim 2. *Every vertex in R either dominates D_x or D_y.*

Proof of Claim: Suppose $w \in R$ does not dominate D_x or D_y. Since w is not dominated by D, we see that $\{x, y, w\}$ is irredundant, a contradiction to the maximality of D.

Claim 3. $\gamma(G[R]) \geq 2$.

Proof of Claim: Suppose that $\gamma(G[R]) = 1$. Let $w \in R$ dominate $G[R]$. By Claim 2, w dominates D_x or D_y. Without loss of generality, assume w dominates D_x. In this case, $\{w, y\}$ is a dominating set of G, a contradiction.

Claim 4. *There exists $w \in R$ that dominates D_x but not D_y and $z \in R$ that dominates D_y but not D_x.*

Proof of Claim: Without loss of generality, suppose every vertex in R dominates D_x. Consider $\gamma(G[D_x])$. First, assume that $\gamma(G[D_x]) = 1$. In this case, if $x' \in D_x$ dominates $G[D_x]$, then $\{x', y\}$ is a dominating set of G, a contradiction. Thus, $\gamma(G[D_x]) \geq 2$. This, implies that $|D_x| > 1$. Let $x' \in D_x$ and $x'' \in D_x$ be non-adjacent. Consider x'. Since we are assuming every vertex in R dominates D_x, we see that x' dominates R. If x' also dominates D_y, then $\{x, x'\}$ dominates all of G, a contradiction. Thus, there exists $y' \in D_y$ such that $x'y' \notin E(G)$. However, we now see that $\{x, y\}$ is not maximal irredundant, since $\{x, x', y\}$ is irredundant (x has x'' as a private neighbor, y has y' as a private neighbor, and x' has any vertex in R as a private neighbor). Thus, our result follows.

Next, we consider the set D_{xy}.

Claim 5. *No vertex in D_{xy} dominates R.*

Proof of Claim: Suppose $v \in D_{xy}$ dominates R. Recall that $deg(v) \leq n - 3$. Since v is adjacent to x, y, and every vertex in R, this implies that there are at least two vertices in $D_x \cup D_y \cup D_{xy}$ that are not neighbors of v. Suppose the only vertices not adjacent to v are in D_x and D_{xy}. This implies that v dominates D_y in which case $\{x, v\}$ is a dominating set, a contradiction. By similar reasoning, the vertices not adjacent to v cannot lie in only D_y and D_{xy}, only D_x, only D_y, or only D_{xy}. Thus, there exists $x' \in D_x$ and $y' \in D_y$ for which $x'v \notin E(G)$ and $y'v \notin E(G)$. This, however, implies that $D = \{x, y\}$ is not maximal irredundant, since $\{x, y, v\}$ is irredundant, a contradiction.

Claim 6. *$D_{xy} \neq \emptyset$.*

Proof of Claim: Suppose that $D_{xy} = \emptyset$. If $\gamma(G[D_x]) = \gamma(G[D_y]) = 1$, with $x' \in D_x$ dominating D_x and $y' \in D_y$ dominating D_y, then, by Claim 2, $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, either $\gamma(G[D_x]) \geq 2$ or $\gamma(G[D_y]) \geq 2$. Without loss of generality, assume that $\gamma(G[D_x]) \geq 2$. Let $x' \in D_x$. If x' does not dominate D_y, then $\{x, y, x'\}$ is irredundant, a contradiction. Hence, we see that every vertex in D_x dominates D_y. This implies that every vertex in D_y dominates D_x as well. Hence, if $x' \in D_x$ and $y' \in D_y$, then $\{x', y'\}$ is a dominating set of G, a contradiction.

Corollary 1. *If G is an isolate free graph on six vertices and has a unique ir-set of cardinality 2, then $\gamma(G) = 2$.*

Finally, we consider the sets D_x and D_y.

Claim 7. *If $v \in D_x$ or $v \in D_y$, then $deg(v) \leq n - 4$.*

Proof of Claim: Since $\Delta(G) \leq n-3$, we see that if $x' \in D_x$, then $deg(x') \leq n-3$. For the sake of contradiction, suppose that $deg(x') = n-3$ and that x' is not adjacent to a and y. (We note that $x' \in D_x$ implies x' is not adjacent to y.)

- If $a \in D_{xy}$ or $a \in D_y$, then $\{x', y\}$ dominates G, a contradiction.
- If $a \in D_x$, then $\{x', x\}$ dominates G, a contradiction.
- If $a \in R$, then since every vertex in R either dominates D_x or D_y, we see that a dominates D_y. This however, implies that $\{x', y'\}$ dominates G for any $y' \in D_y$.

Hence, we have arrived at a contradiction. Our claim follows.

Claim 8. *If $x' \in D_x$, then x' either dominates D_x or D_y. If $y' \in D_y$, then y' either dominates D_x or D_y.*

Proof of Claim: If $x' \in D_x$ does not dominate D_x and does not dominate D_y, then $\{x, x', y\}$ is irredundant by Claim 4.

Corollary 2. *If $\gamma(G[D_x]) > 1$ or $\gamma(G[D_y]) > 1$, then each vertex in D_x dominates D_y and each vertex in D_y dominates D_x.*

Proof. Suppose $\gamma(G[D_x]) > 1$. This implies that no vertex in D_x dominates D_x. Hence, by Claim 8, each vertex in D_x dominates D_y. This also implies that each vertex in D_y dominates D_x. The case of $\gamma(G[D_y]) > 1$ follows similarly. □

Claim 9. *No vertex in D_x or D_y dominates D_{xy}.*

Proof of Claim: Suppose $x' \in D_x$ dominates D_{xy}. x' itself either dominates D_x or D_y by Claim 8. We consider each case.

First suppose x' dominates D_x. If there exists $y' \in D_y$ that dominates D_y, then $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, no vertex in D_y dominates D_y. Hence, every vertex in D_y dominates D_x. This, however, implies that every vertex in D_x also dominates D_y. Hence, $\{x', y'\}$ is a dominating set of G for any $y' \in D_y$, a contradiction.

Suppose now that x' dominates D_y but not D_x. If there exists $y' \in D_y$ such that y' dominates D_x then $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, no vertex in D_y dominates D_x, in which case every vertex in D_y dominates D_y. Let x'' be a vertex in D_x not dominated by x'. If x'' does not dominate D_y, then $\{x, x'', y\}$ is irredundant, a contradiction. Thus, x'' dominates D_y. Since this is true for every vertex in D_x not dominated by x', we see that each vertex in D_y dominates the vertices in D_x not dominated by x'. This implies that $\{x', y'\}$ is a dominating set of G for any $y' \in D_y$.

Claim 10. *No vertex in D_x dominates R. No vertex in D_y dominates R.*

Proof of Claim: Suppose $x' \in D_x$ dominates R. If x' dominates D_x, then $\{x', y\}$ dominates G, a contradiction. Thus, x' dominates D_y. This, however, implies that $\{x, x'\}$ dominates G, again a contradiction. Hence, no vertex in D_x dominates R. By the same logic, no vertex in D_y dominates R.

Claim 11. *For each pair of vertices $\{x', y'\}$ such that $x' \in D_x$ and $y' \in D_y$, there exists a vertex $v \in D_{xy}$ not adjacent to either x' or y'.*

Proof of Claim: Suppose $x' \in D_x$, $y' \in D_y$, and that $D_{xy} \subseteq (N[x'] \cup N[y'])$. We consider several cases.

- If x' dominates D_x and y' dominates D_y, then $\{x', y'\}$ dominates G, a contradiction.
- If x' dominates D_y and y' dominates D_x, then $\{x', y'\}$ dominates G, a contradiction.
- Suppose both x' and y' dominate D_y but not D_x. In this case, consider $x'' \in D_x - N[x']$. If x'' does not dominate D_y, then $\{x, x'', y\}$ is irredundant, a contradiction. Thus, we see that each vertex in $D_x - N[x']$ dominates D_y, in which case every vertex in D_y dominates $D_x - N[x']$. Thus, once again we see that $\{x', y'\}$ dominates all of G, a contradiction.
- If both x' and y' dominate D_x but not D_y, then $\{x', y'\}$ will dominate all of G, in a manner similar to the case above.

Our result follows.

Considering the results above, we see that G has one of the four graphs below as an induced subgraph.

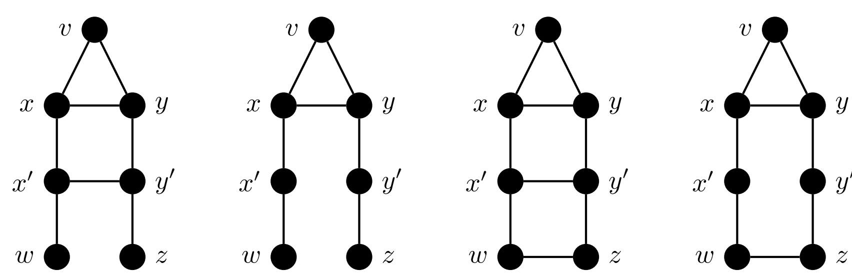

Figure 2: Induced Subgraphs

We now show that if $|R| \geq 3$, $|D_x| \geq 2$, or if $|D_y| \geq 2$, then $|E(G)| \leq |E(F)| = \binom{n-2}{2} - 2$, where F is the graph considered at the beginning of this section. We prove this by constructing a graph G' from G satisfying $|E(G)| \leq |E(G')|$ and $G' \subseteq F$.

Suppose that at least one of the following is true concerning G.

- $|R| \geq 3$
- $|D_x| \geq 2$
- $|D_y| \geq 2$

Find, and label, vertices w and z in R such that w dominates D_x but not D_y and such that z dominates D_y but not D_x. Next, find, and label, vertex x' in D_x that is not dominated by z and vertex y' in D_y that is not dominated by w. Finally, for the pair $\{x', y'\}$, find, and label, the vertex v in D_{xy} that is not dominated by x' or y'. Observe that v is not adjacent to w or z (if v is adjacent to either w or z, then $\{x, y, v\}$ is irredundant). Define the following sets.

- $D_x^* = D_x - \{x'\}$
- $D_y^* = D_y - \{y'\}$
- $R^* = R - \{w, z\}$
- $D_{xy}^* = D_{xy} - \{v\}$

Note that if there are no edges between $\{w, z\}$ and D_x^*, D_y^*, D_{xy}^*, and R^*, then G is isomorphic to a subgraph of F above. Bearing this in mind, consider the following procedure.

Let G' be a distinct copy of G. We alter G' after considering G.

1. If $wz \notin E(G)$, continue to Step 2 below. If $wz \in E(G)$, then there exists $r \in R^*$ for which $wr \notin E(G)$ by Claim 3. In G and G', delete the edge wz and add the edge wr.

2. Consider D_x^* in G. If $|D_x^*| = 0$, continue to Step 3 below. Otherwise, for each $s \in D_x^*$ proceed as follows. Note that s is adjacent to w, but it is not adjacent to y (by definition of D_x) and it is not adjacent to at least one vertex in R (by Claim 10), call it r_s. Delete the edge sw and add the edge sy in G'. Additionally, if s is adjacent to z, delete the edge sz and add the edge sr_s in G'. Note that after the completion of Step 2, $|E(G)| = |E(G')|$ and $N_{G'}(\{w, z\}) \cap D_x^* = \emptyset$.

3. Consider D_y^* in G. If $|D_y^*| = 0$, continue to Step 4 below. Otherwise, for each $t \in D_y^*$, proceed as follows. Note that t is adjacent to z, but is not adjacent to x (by definition of D_y) and it is not adjacent to at least one vertex in R (by Claim 10), call it r_t. In G', delete the edge tz and add the edge tx. Additionally, if t is adjacent to w, in G' delete the edge tw and add the edge tr_t. Note that after the completion of Step 3, $|E(G)| = |E(G')|$ and $N_{G'}(\{w, z\}) \cap D_y^* = \emptyset$.

4. Consider R^* in G. If $|R^*| = 0$, continue to Step 5 below. Otherwise, for each $u \in R^*$, proceed as follows. Note that u is not adjacent to x, y, or v, or else $\{x, y, v\}$ is irredundant. Add the edges uv, ux, and uy in G'. If u is adjacent to w in G, delete the edge uw from G'. If u is adjacent to z in G, delete the edge uz from G'. Note that after the completion of Step 4, $|E(G)| \leq |E(G')|$ and $N_{G'}(\{w, z\}) \cap R^* = \emptyset$.

5. Finally, consider D_{xy}^* in G. If $|D_{xy}^*| = 0$, then we're done. Otherwise, partition D_{xy}^* as follows.

   - Let S_{1A} denote the set of vertices p in D_{xy}^* which dominate all but one vertex in R, and which do not dominate $D_x \cup D_y \cup \{v\}$.
   - Let S_{1B} denote the set of vertices p in D_{xy}^* which dominate all but one vertex in R, and which dominate $D_x \cup D_y \cup \{v\}$.
   - Let S_2 denote the set of vertices p in D_{xy}^* which do not dominate two or more vertices in R.

   For each $p \in S_{1A}$, proceed as follows. Let o_p denote the vertex in $D_x \cup D_y \cup \{v\}$ that p is not adjacent to, and let r_p denote the vertex in R that p is not adjacent to. In G', delete the edges pw and pz (if they exist), add the edge po_p, and add the edge pr_p if and only if r_p is distinct from w and z.

   For each $p \in S_{1B}$, proceed as follows. Let r_p denote the vertex in R that p is not adjacent to. First, observe that there is at least one vertex in $D_{xy} - N[p]$ since $deg(p) \leq n - 3$. Next, note that $(D_{xy} - N[p]) \cap (S_{1A} \cup S_2) \neq \emptyset$, since otherwise $\{p, x'\}$ or $\{p, y'\}$ dominates G (depending upon whether r_p dominates D_x or D_y). Thus, let $o_p \in (D_{xy} - N[p]) \cap (S_{1A} \cup S_2)$. In G', delete the edges pw and pz (if they exist), add the edge po_p, and add the edge pr_p if and only if r_p is distinct from w and z.

   For each $p \in S_2$, let r_1 and r_2 denote the vertices in R that p is not adjacent to. In G', delete the edges pw and pz (if they exist), and add the edges pr_1 and pr_2 if and only if r_1 and r_2 are distinct from

w and z respectively.

Note that after the completion of Step 5, $|E(G)| = |E(G')|$ and that $N_{G'}(\{w, z\}) \cap D^*_{xy} = \emptyset$.

Upon completion of Step 5, we see that the graph G' is isomorphic to a subgraph of the graph F constructed at the beginning of this section, and that $|E(G)| \leq |E(G')|$. Hence, we have proven that if $|R| > 2$, $|D_x| > 1$, or $|D_y| > 1$, then $|E(G)| \leq \binom{n-2}{2} - 2$.

Suppose now that G satisfies $|R| = 2$, $|D_x| = 1$, $|D_y| = 1$ and $|D_{xy}| = n-6$. Find, and label, the vertices w, z, x', y' and v as before. Note that x' is not adjacent to z and that y' is not adjacent to w by Claim 10. By Claim 3, w is not adjacent to z. Additionally, v is not adjacent to either w or z, since otherwise $\{x, y, v\}$ is irredundant. If no vertex in D_{xy} shares an edge with w or z, then G is a subgraph of F, in which case $|E(G)| \leq \binom{n-2}{2} - 2$. Thus, suppose there exists a vertex, call it p, in D_{xy} which is adjacent to a vertex in R. Note that $|N(p) \cap R| \leq 1$ by Claim 5. Create a copy G' of G, and proceed as follows.

1. For each vertex $p \in D_{xy}$, if p is adjacent to a vertex in R but is not adjacent to one of x', y', or v, delete the edge from p to R in G' and add the edge px', py', or pv as appropriate.

2. If there are still vertices in D_{xy} that share an edge with a vertex in R, proceed as follows. Suppose that $p \in D_{xy}$ is adjacent to a vertex in R, say z without loss of generality. Consider then x'. Note that x' does not dominate $V(G) - N[p]$ since otherwise $\{x', p\}$ dominates G. Hence, there exists a vertex which neither x' nor p is adjacent to, call it c (note that $c \neq z$). In G', exchange the pz edge for the pc edge. Since $x'c \notin E(G)$, c does not share an edge with a vertex in R.

Since the total number of edges is preserved in Step 1, and since the only edges added in Step 2 are from a vertex sharing an edge with a vertex in R to a vertex not sharing an edge with a vertex in R, we see that $|E(G)| = |E(G')|$. Since G' is a subgraph of F, we see that $|E(G)| \leq \binom{n-2}{2} - 2$. We have thus proven our result. $\square$

4.2 D is a dominating set

Suppose now that G is an isolate free graph on n-vertices ($n \geq 6$) having a unique ir-set of cardinality 2, call it D, which is a dominating set. Since D is a dominating set, this implies that D is a γ-set, since if $\gamma(G) = 1$, then $ir(G) = 1$ as well, a contradiction. Moreover, D is a unique γ-set in G, since if G has a γ-set distinct from D, call it D', then D' is maximal irredundant, contradicting the uniqueness of D. Hence, by Theorem 3, $|E(G)| \leq \binom{n-2}{2}$. To see that this bound can be achieved, consider the following two constructions.

First, the following graph on six vertices, together with Corollary 1, shows that $m_{ir}(6, 2) = \binom{6-2}{2} = 6$.

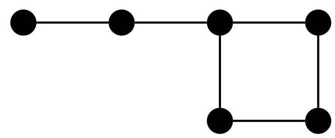

Figure 3: $n = 6$ case

For the $n \geq 7$ case, consider the following. Let

$$V(H) = \{x_1, x_2, \ldots, x_{n-5}, y_1, y_2, s, l, p\}.$$

Let $H[\{x_1, x_2, \ldots, x_{n-5}\}]$ be complete. Additionally, let

$$\begin{aligned} N(y_1) &= \{x_1, x_2, \ldots, x_{n-5}, s\} \\ N(y_2) &= \{x_1, x_2, \ldots, x_{n-5}, s, p\} \\ N(s) &= \{y_1, y_2\} \\ N(l) &= \{x_1\} \\ N(p) &= \{x_2, x_3, \ldots, x_{n-5}, y_2\}. \end{aligned}$$

The case of $n = 7$ is illustrated in the figure below.

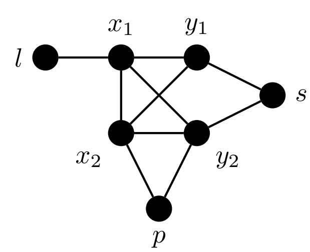

Figure 4: $n = 7$ case

Similarly to the graph F considered in the previous section, the reader can verify that H has a unique ir-set of cardinality 2 given by $\{x_1, y_2\}$. Since $|E(H)| = \binom{n-2}{2}$, we have the following result.

Theorem 5. *For $n \geq 6$, $m_{ir}(n,2) = \binom{n-2}{2}$.*

Additionally, by considering our work at the beginning of this section, we have the following.

Theorem 6. *For $n \geq 4$, $m^*_{ir}(n,2) = \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$.*

Before concluding this section, we note that not every graph G having a unique γ-set of cardinality 2 has a unique ir-set of cardinality 2, even when $ir(G) = \gamma(G) = 2$. For example, the graph P_6 has a unique γ-set of cardinality 2, but does not have a unique ir-set. Moreover, the set of isolate free graphs on n vertices having a unique ir-set of cardinality 2 and a maximum number of edges is a *proper* subset of the set of all isolate free graphs on n vertices having a unique γ-set of cardinality 2 and a maximum number of edges. For example, the graph in Figure 5 below has a unique γ-set of cardinality 2 (and a maximum number of edges), but does not have a unique ir-set, even though $ir(G) = 2$.

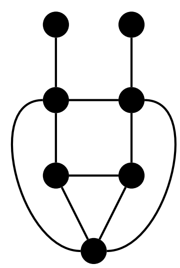

Figure 5: $\gamma(G) = ir(G) = 2$, unique γ-set, not a unique ir-set

5 Unique minimum independent dominating sets

In this section, we consider the maximum number of edges in a graph on n vertices having a unique i-set of cardinality 2. First, we note that no graph on one or three vertices has a unique i-set of cardinality 2. The only graph on two vertices having a unique i-set of cardinality two is the completely disconnected graph $\overline{K_2}$. Additionally, the only graph on four vertices which has a unique i-set of cardinality two is $P_3 \cup K_1$. Putting these trivial cases aside, we now restrict ourselves to graphs on at least five vertices. Let G be a graph on $n \geq 5$ vertices having a unique minimum independent dominating set of cardinality 2 and having a maximum number of edges. Let D denote the unique i-set of G, and for notational purposes, let $D = \{x, y\}$. Additionally, let $R = V(G) - D$. Define the following sets.

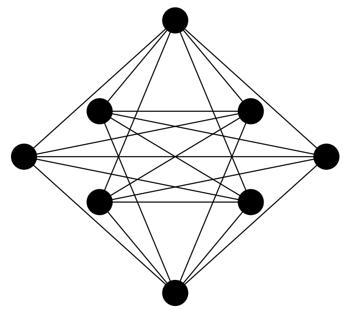

Figure 6: $m_i(8,2) = 21$

- $D_x = N(x) - N(y)$
- $D_y = N(y) - N(x)$
- $D_{xy} = N(x) \cap N(y)$.

Consider a vertex $v \in R$. What is the maximum degree of v? We see that if $deg(v) = n-1$, then $\{v\}$ itself is an independent dominating set of cardinality one, a contradiction. If $deg(v) = n-2$, with v not adjacent to u, then $\{u, v\}$ is an independent dominating set of cardinality two distinct from D, a contradiction. Thus, $deg(v) \leq n-3$. Additionally, note that if $deg(v) = n-3$, then the two vertices not adjacent to v are not adjacent.

Using the familiar result that $|E(G)| = \frac{1}{2}\sum_{v \in V(G)} deg(v)$, we see that

$$\begin{aligned}
|E(G)| &\leq \frac{1}{2}\left(|D_x| + 2|D_{xy}| + |D_y| + (n-3)(n-2)\right) \\
&\leq \frac{1}{2}\left(2(n-2) + (n-3)(n-2)\right) \\
&= \frac{1}{2}\left((n-2)(n-1)\right) \\
&= \binom{n-1}{2}.
\end{aligned}$$

When $n = 3k+2$, $k \geq 1$, this upper bound on $|E(G)|$ can be achieved if $R = D_{xy}$ and $G[R]$ is a complete graph minus the edges of k disjoint triangles (see Figure 6). Thus, we see that $m_i(3k+2, 2) = m_i^*(3k+2, 2) = \binom{3k+1}{2}$.

Suppose now that $n = 3k$ or $n = 3k+1$, for $k \geq 2$. Note that the upper bound on $|E(G)|$ found above is achievable if and only if $R = D_{xy}$ and if each vertex $v \in R$ satisfies $deg(v) = n-3$. Thus, suppose that $R = D_{xy}$ and that each vertex in R has degree $n-3$. If $v \in R$ is not adjacent to vertices v_1 and v_2, by our observations above, $v_1v_2 \notin E(G)$. Since we are assuming $R = D_{xy}$, we see that $v_1 \in D_{xy}$ and $v_2 \in D_{xy}$. Hence, $deg(v_1) = n-3$ and $deg(v_2) = n-3$. Moreover, the two vertices not adjacent to v_1 are v and v_2, while the two vertices not adjacent to v_2 are v and v_1. Thus, each of v, v_1 and v_2 dominates $R - \{v, v_1, v_2\}$. Since the above logic can be applied to each vertex in R, R can be partitioned into sets of cardinality 3 such that each set S induces an independent set in G that dominates $R - S$. This, however, is clearly a contradiction since $|R| \not\equiv 0 \pmod 3$. Thus, if $n = 3k$ or $n = 3k+1$, we have $|E(G)| \leq \binom{n-1}{2} - 1$. This upper bound can be achieved in each case as follows.

If $n = 3k$, we let $R = D_{xy}$. Initially, we let $G[R]$ be complete. After removing the edges of $k-1$ disjoint triangles from R, find the one remaining vertex in R of degree $n-1$ and make it not adjacent to any two other vertices in R. An example construction is illustrated in Figure 7.

If $n = 3k+1$, we once again let $R = D_{xy}$ and once again initially set $G[R]$ to be complete. In this case, after removing the edges of $k-1$ disjoint triangles from R, we find the remaining two vertices in R of degree $n-1$, call them v_1 and v_2. We remove the edge v_1v_2 from G, and then pick an arbitrary vertex $v \in R$, ($v \neq v_1$ and $v \neq v_2$) and remove the edges vv_1 and vv_2. An example construction is illustrated in Figure 7.

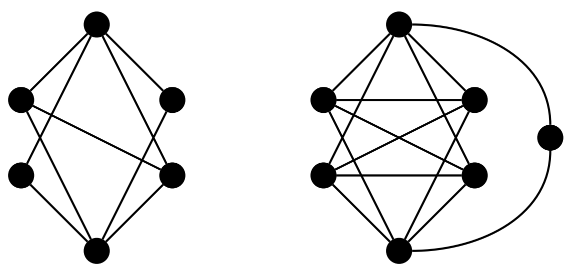

Figure 7: $m_i(6,2) = 9$ and $m_i(7,2) = 14$

Thus, we summarize our results as follows.

Theorem 7. *For $n = 2$, $n = 4$, or $n \geq 5$,*

$$m_i^*(n,2) = \begin{cases} \binom{n-1}{2} & \text{if } n \equiv 2 \pmod 3, \\ \binom{n-1}{2} - 1 & \text{otherwise.} \end{cases}$$

By observing that the graphs constructed in the $n \geq 5$ case are all isolate free, we also have the following.

Corollary 3. *For $n \geq 5$,*

$$m_i(n,2) = \begin{cases} \binom{n-1}{2} & \text{if } n \equiv 2 \pmod 3, \\ \binom{n-1}{2} - 1 & \text{otherwise.} \end{cases}$$

6 Unique maximum independent, maximum minimal dominating, and maximum irredundant sets

In this section, we consider unique β_0-sets, Γ-sets, and IR-sets of cardinalities at least 2. We begin with unique β_0-sets.

Theorem 8. *Let $k \geq 2$. For $n \geq k$, $m_{\beta_0}(n,k) = m^*_{\beta_0}(n,k) = \binom{n}{2} - \binom{k}{2}$.*

Proof. Let $k \geq 2$, and let G be a graph on $n \geq k$ vertices having a unique β_0-set of cardinality k. First, observe that since G contains k mutually non-adjacent vertices, $|E(G)| \leq \binom{n}{2} - \binom{k}{2}$. Since the graph $K_n - E(K_k)$ has a unique β_0-set of cardinality k and has $\binom{n}{2} - \binom{k}{2}$ edges, our result follows. $\square$

As it turns out, unique Γ-sets and unique IR-sets of cardinality k can be handled similarly.

Theorem 9. *Let $k \geq 2$. For $n \geq k$,*

$$m_\Gamma(n,k) = m^*_\Gamma(n,k) = m_{IR}(n,k) = m^*_{IR}(n,k) = \binom{n}{2} - \binom{k}{2}.$$

Proof. Let $k \geq 2$, and let G be a graph on $n \geq k$ vertices having a unique Γ-set or a unique IR-set of cardinality k, call it D. In either case, we have that D is maximal irredundant.

Partition D into two subsets S_1 and S_2 such that every vertex in S_1 has an external private neighbor, while every vertex in S_2 does not have an external private neighbor. If $S_1 = \emptyset$, then D is an independent set in which case $|E(G)| \leq \binom{n}{2} - \binom{k}{2}$ as illustrated in the proof of Theorem 8.

Thus, suppose that $S_1 \neq \emptyset$. Note that for each vertex v in S_1, there is a vertex $u_v \in V(G) - D$ satisfying $N[u_v] \cap D = \{v\}$. We construct a graph G' from G as follows. For each $v \in S_1$, let the vertices in $N(v) \cap D$ be $w_1, w_2, \ldots, w_r$. Note that $w_1, w_2, \ldots, w_r \in S_1$, since if $w_j \in S_2$, then w_j does not have a private neighbor by definition of S_2. Delete the edge vw_i and add the edges $w_i u_v$ and vu_{w_i} for each $1 \leq i \leq r$. If $N(v) \cap D = \emptyset$, no edges need to be deleted or added when considering v. Once this has been completed, we see that D forms an independent set in G'. Since $|E(G')| \geq |E(G)|$, we see that $|E(G)| \leq \binom{n}{2} - \binom{k}{2}$.

The graph $K_n - E(K_k)$ satisfies $\Gamma(K_n - E(K_k)) = IR(K_n - E(K_k)) = k$ and has a unique Γ-set and a unique IR-set. Thus, our result follows. $\square$

7 Concluding Remarks and Open Problems

By combining Proposition 2 and Theorems 3, 5, 6, 7, 8, and 9, we have our main result.

Theorem. *For $n \geq 6$*

$$m_{ir}(n,2) = m_{\gamma}(n,2) \leq m_i(n,2) \leq m_{\beta_0}(n,2) = m_{\Gamma}(n,2) = m_{IR}(n,2)$$

and

$$m^*_{ir}(n,2) = m^*_{\gamma}(n,2) \leq m^*_i(n,2) \leq m^*_{\beta_0}(n,2) = m^*_{\Gamma}(n,2) = m^*_{IR}(n,2).$$

Before concluding our work, we pose the following open problems.

- What are the values for $m_{ir}(n,k)$ and $m^*_{ir}(n,k)$ for $k \geq 3$?
- What are the values for $m_i(n,k)$ and $m^*_i(n,k)$ for $k \geq 3$?
- For $k \geq 3$ and n sufficiently large, do the following inequalities hold?

  $$m_{ir}(n,k) \leq m_{\gamma}(n,k) \leq m_i(n,k) \leq m_{\beta_0}(n,k) = m_{\Gamma}(n,k) = m_{IR}(n,k)$$

  and

  $$m^*_{ir}(n,k) \leq m^*_{\gamma}(n,k) \leq m^*_i(n,k) \leq m^*_{\beta_0}(n,k) = m^*_{\Gamma}(n,k) = m^*_{IR}(n,k).$$

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