The Kings Problem and Matrix Recurrences

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1 Introduction

Let an $n \times k$ chessboard be given. The kings problem asks for the number of ways to place kings on it so that no two are adjacent. This amounts to finding all the configurations of the chessboard that do not include any of the following,

K	
K	

K	K

K	
	K

	K
K	

The special case where $k = 1$ is completely solved. Let an $n \times 1$ board be given, and let $x_1(n) :=$ the number of allowed configurations that do not have a king in the last cell. Let $x_2(n) :=$ the number of allowed configurations that do. Then we get

$$x_1(n) = x_2(n-1) + x_1(n-1) = x_1(n-2) + x_1(n-1)$$

and

$$x_2(n) = x_1(n-1) = x_2(n-2) + x_1(n-2) = x_2(n-3) + x_2(n-2)$$

The sum $x_1(n) + x_2(n)$ is also a Fibonacci sequence in n, appropriately offset

Let us modify this approach so that we can to generalize to higher dimensions. We summarize the relationship above by the following matrix:

$$\begin{pmatrix} x_1(n-1) & x_2(n-1) \\ x_1(n-2) & x_2(n-2) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}$$

Putting in the starting conditions for $n = 0$ and $n = 1$, we get a sequence in n of vectors defined explicitly by

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \mathbf{x}(n+1)$$

We should examine more closely what the ones and zeroes mean in this matrix. There are two possible combinations for the last "column" of an $n \times 1$ board. If the column is empty, let us say that it has configuration x_1. Else, let us say that it has configuration x_2. If the $(n+1)^{st}$ cell has configuration x_1, then we can place it beside a cell that has either configuration x_1 or configuration x_2. That means we need both of those to contribute to the top part of the output vector, $x_1(n-1)$. We put a one in row 1, column 1 of the matrix to express that configuration x_1 can be placed beside itself. We also put a one in row 1, column 2 to express that configuration x_1 can be put next to configuration x_2. Matters are different when the $(n+1)^{st}$ cell has configuration x_2, a king. That cell can be put next to an empty cell, but not next to another king. Then, there are contributions to $x_2(n-1)$ from only one place in the matrix. Let us put that one in column 1, to mean that we can place combination x_2 beside combination x_1. The matrix we have just constructed has a dual nature as an adjacency matrix of a graph that has a node for every possible combination x_1 and x_2.

Let us construct a similar adjacency matrix for the $n \times k$ case. We will list all possible configurations for the last column in a very specific order. First, we list the columns with the bottom-most square empty, then the columns with the bottom-most square full. For each of these two groups, start with the columns that have the second square from the bottom empty. For each of the four resulting groups, start with the ones with the third bottom-most square empty, etc. Continue moving up until all the combinations are listed. The list for the $n \times 4$ board is shown in figure 1.

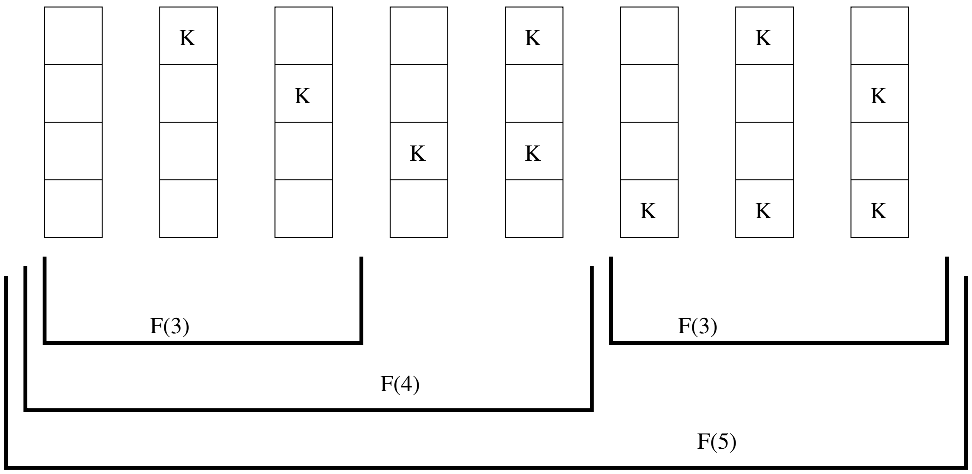

Figure 1: Listing these columns is isomorphic to listing rows of a 4×1 board.

To form an adjacency we need to know which columns can be placed side by side. Clearly, if the bottom squares in both columns are empty, we are just placing columns of $n \times k - 1$ board. Then, the top left corner of A_k is simply A_{k-1}. Next we try to place the columns in figure 1. Since kings are not allowed to be

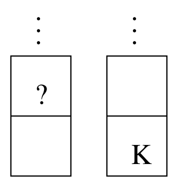

Figure 2: Possible final positions of a column

placed diagonally next to each other, the position corresponding to a question mark figure 1 must be blank. Then, the position above it can be marked with anything we choose, so we are essentially placing columns of an $n \times (k-2)$ board. By the order of the columns, this corresponds to putting A_{k-2} immediately to the bottom right of A_{k-1}. Since column placement is symmetric, that is if we can place a column to the right of another column we can also place it to the left, the matrix is symmetric. Thus we need place another A_{k-2} immediately to the top left of A_{n-1}. Since you are neither allowed to place two columns that both have a king in the bottom position side by side, nor are you allowed to place two columns that have a king in the last and second to last positions side by side, the remaining squares of A_n are filled with zeros.

More precisely, we obtain the matrix recurrence,

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-2} \\ A_{n-2} & 0 \end{pmatrix}$$

2 A Brief Overview of Tensors

Define the operation $\otimes$ to be the Kronecker tensor. In other words $\otimes$ is a well defined mapping from two arbitrary matrices $A \in Lin(\mathbb{F}^m), B \in Lin(\mathbb{F}^n)$ to the matrix $A \otimes B \in \mathbb{F}^{mn}$ such that,

$$A = \begin{bmatrix} a_{0,0} & \dots & a_{0,j-1} \\ \vdots & \ddots & \vdots \\ a_{i-1,0} & \dots & a_{i-1,j-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{0,0} & \dots & b_{0,c-1} \\ \vdots & \ddots & \vdots \\ b_{r-1,0} & \dots & b_{r-1,c-1} \end{bmatrix}$$

then

$$A \otimes B = \begin{bmatrix} a_{0,0}B & \dots & a_{0,j-1}B \\ \vdots & \ddots & \vdots \\ a_{i-1,0}B & \dots & a_{i-1,j-1}B \end{bmatrix}$$

More specifically

$$A \otimes B = \begin{bmatrix} a_{0,0}b_{0,0} & \dots & a_{0,0}b_{0,c-1} & & a_{0,j-1}b_{0,0} & \dots & a_{0,j-1}b_{0,c-1} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{0,0}b_{r-1,0} & \dots & a_{0,0}b_{r-1,c-1} & & a_{0,j-1}b_{r-1,0} & \dots & a_{0,j-1}b_{r-1,c-1} \\ & \vdots & & \ddots & & \vdots & \\ a_{i-1,0}b_{0,0} & \dots & a_{i-1,0}b_{0,c-1} & & a_{i-1,j-1}b_{0,0} & \dots & a_{i-1,j-1}b_{0,c-1} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{i-1,0}b_{r-1,0} & \dots & a_{i-1,0}b_{r-1,c-1} & & a_{i-1,j-1}b_{r-1,0} & \dots & a_{i-1,j-1}b_{r-1,c-1} \end{bmatrix}$$

Under this operation the following relations hold [3]. Suppose

$$\dim(A) = \dim(C) \quad \text{and} \quad \dim(B) = \dim(D)$$

then

$$\begin{aligned} A \otimes B + A \otimes D &= A \otimes (B + D) \\ A \otimes B + C \otimes B &= (A + C) \otimes B \\ (A \otimes B)(C \otimes D) &= (AC \otimes BD) \end{aligned}$$

Additionally for all $c \in \mathbb{F}$ and matrices A and B over $\mathbb{F}$

$$c(A \otimes B) = cA \otimes B = Ac \otimes B = A \otimes cB = A \otimes Bc = (A \otimes B)c$$

Let $J_n(\lambda)$ denote a Jordan block of size n with λ along the diagonal. Let the matrices A and B have Jordan forms

$$\bigoplus_{1 \le k \le s} (J_{n_k}(\alpha_k)) \quad \text{and} \quad \bigoplus_{1 \le r \le t} (J_{m_r}(\beta_r)) \quad \text{respectively}$$

then $A \otimes B$ is similar to

$$\bigoplus_{1 \le k \le s, 1 \le r \le t} (J_{n_k}(\alpha_k) \otimes J_{m_r}(\beta_r))$$

and in general [1]

$$Jordan(A \otimes B) = Jordan(B \otimes A)$$

To determine the structure of the tensor product of two arbitrary matrices, it sufficies for us to show the Jordan structure of the tensor of two Jordan blocks. Let n and m be positive integers, with $n \le m$. Let $\alpha, \beta \ne 0$ then

$$Jordan(J_n(0) \otimes J_m(0)) = \big(\bigoplus_{i=1}^{n-1} J_i(0) \oplus J_i(0)\big) \oplus (J_n(0) \otimes I_{m-n+1}) \tag{1}$$

$$Jordan(J_n(\alpha) \otimes J_m(0)) = \bigoplus_{j=1}^{n} J_m(0) \tag{2}$$

similarly

$$Jordan(J_n(0) \otimes J_m(\beta)) = \bigoplus_{j=1}^{m} J_n(0) \tag{3}$$

and

$$Jordan(J_n(\alpha) \otimes J_m(\beta)) = \bigoplus_{k=0}^{n-1} J_{m-n+1+2k}(\alpha\beta) \tag{4}$$

Proofs of equations $1, 2, 3, 4$ have appeared in Li[1].

3 Diagonalizable Recurrence

Theorem 3.1 *Let α and β be diagonalizable $n \times n$ matrices, say*

$$P_1 \alpha P_1^{-1} = D_1$$
$$P_2 \beta P_2^{-1} = D_2$$

Then $\alpha \otimes \beta$ is diagonalizable.

Proof

$$\begin{aligned}(P_1 \otimes P_2)(\alpha \otimes \beta)(P_1 \otimes P_2)^{-1} &= (P_1 \otimes P_2)(\alpha P_1^{-1} \otimes \beta P_2^{-1}) \\ &= (P_1 \alpha P_1^{-1} \otimes P_2 \beta P_2^{-1}) \\ &= (D_1 \otimes D_2)\end{aligned}$$

□

Corollary 3.1 *Let α be a diagonalizable 2×2 matrix and let $A_n = \alpha^{\otimes n}$. If A_1 has eigenvalues $\lambda_1 > \lambda_2$ then A_n has eigenvalues $\lambda_1^i \lambda_2^{n-i}$ with multiplicity $\binom{n}{i}$.*

Proof
Proof by induction on n. A_1 is diagonalizable by definition. Assume A_{n-1} is diagonalizable. Then, A_{n-1} has eigenvalues $\lambda_1^i \lambda_2^{n-i-1}$ with multiplicity $\binom{n-1}{i}$. Now, $\alpha \otimes A_{n-1}$ has eigenvalues $\lambda_1 \lambda_1^i \lambda_2^{n-i-1}$ with multiplicity $\binom{n-1}{i}$ and $\lambda_2 \lambda_1^i \lambda_2^{n-i-1}$ with multiplicity $\binom{n-1}{i}$ by the definition of the Kronecker Tensor. So, the eigenvalue $\lambda_1^i \lambda_2^{n-i}$ has multiplicity $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$.

□

Definition 3.1 *A* **double matrix recurence** *is a recurence of the form*

$$A_n = \alpha \otimes A_{n-1} + \beta \otimes B_{n-1}$$
$$B_n = \gamma \otimes A_{n-1} + \delta \otimes B_{n-1}$$

where $\alpha, \beta, \gamma, \delta$ are $m \times m$ matrices.

Theorem 3.2 *If $A_0, B_0, \alpha, \beta, \gamma,$ and δ (as defined in definition 3.1) are all simultaneaously diagonalized by some matrix, say P, then A_n and B_n are diagonalized by $P^{\otimes n}$.*

Proof

Proof by induction on n. Assume that A_{n-1} and B_{n-1} are diagonalized by $P^{\otimes n-1}$. Then,

$$\begin{aligned}
P^{\otimes n}A_n(P^{\otimes n})^{-1} &= P^{\otimes n}\left(\alpha \otimes A_{n-1} + \beta \otimes B_{n-1}\right)(P^{\otimes n})^{-1} \\
&= \left(P \otimes P^{\otimes n-1}\right)\left(\alpha \otimes A_{n-1} + \beta \otimes B_{n-1}\right)\left(P^{-1} \otimes (P^{\otimes n-1})^{-1}\right) \\
&= \left(P\alpha P^{-1}\right) \otimes \left(P^{\otimes(n-1)}A_{n-1}(P^{\otimes(n-1)})^{-1}\right) \\
&\quad + \left(P\beta P^{-1}\right) \otimes \left(P^{\otimes(n-1)}B_{n-1}(P^{\otimes(n-1)})^{-1}\right)
\end{aligned}$$

Since the tensor and sum of two diagonal matrices is again diagonal, A_n is diagonalizable by $P^{\otimes n}$. The proof for B_n follows the same proof as that for A_n.

□

The adjacency matrix for the King's problem is formed by a double matrix recurence. However there is not a matrix P that simultaneously diagonalizes this double martix recurence. For the kings problem take, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

4 Non-negative Matrices

Theorem 4.1 (Perron-Fröebnious) *Given a primitive $n \times n$ matrix T there exists an eigenvalue λ such that [4],*

1 $\lambda \in \mathbb{R}$ and $\lambda > 0$

2 λ has unique left and right eigenvectors up to constant multiples

3 $\lambda > |\lambda^{'}|$ for all eigenvalues $\lambda^{'} \neq \lambda$

4 λ is a simple root of the characteristic equation of T

Li and Strouse [1] offer an elementry proof of the formula

$$Jordan(J_n(\alpha) \otimes J_m(\beta)) = \bigoplus_{k=0}^{n-1} J_{m-n+1+2k}(\alpha\beta) \tag{5}$$

(5) allows us to make an alternative proof for the last part of Perron Fröbenious using only elementry methods. More precisely, given that any non-negative primitive matrix T has a positive real dominant eigenvalue λ with corrosponding eigenspace of dimension one it follows from the formula above that λ is a simple root of the characteristic polynomial for T.

Proof

Let T be a non-negative primitive matrix with dominant eigenvalue λ, then λ^2 is the dominant eigenvalue of $T \otimes T$. Now, since $Jordan(T)$ contains one and only one eigen-block for $\lambda, J_n(\lambda)$ corrosponds to the one dimensional eigenspace for λ. Since T is primitive, T^k is strictly positive, which implies $(T \otimes T)^k = T^k \otimes T^k$ is strictly possitive. Ergo $T \otimes T$ is primitive, with dominant eigen-value λ^2

Now, $Jordan(T \otimes T)$ contains one and only one eigenblock for $\lambda^2, J_m(\lambda^2)$ corrosponding to the one dimensional eigenspace for λ. Also, $Jordan(J_n(\lambda) \otimes J_n(\lambda))$ is a direct summand of $Jordan(T \otimes T)$. Therefore, $Jordan(J_n(\lambda) \otimes J_n(\lambda)) = J_m(\lambda^2)$.

Equation 5 above implies that the the number of Jordan blocks in $Jordan(J_n(\lambda) \otimes J_n(\lambda))$ is n. So $n = 1$ and λ is a simple root of the characteristic polynomial of T.

5 Combinatorial Analysis of a Single Matrix recurrence

Consider a matrix α that is a Jordan block of size j. Define

$$c_{n,j,k} = \# \text{ of Jordan blocks of size } n \text{ in the Jordan Canonical form of } \alpha^{\otimes k}$$

Let us first look at the simple case when α is a Jordan block of size 2. Then, we start with a Jordan block of size 2 at level 1 (ie $k = 1$). Tensoring that with another Jordan block of size 2 we produce a Jordan block of size 3 and 1 by Li[1]. Continuing the process we get the triangle as formed in figure 3.

$k \diagdown n$	1	2	3	4	5	6	7	8	9
1	0	1	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0
3	0	2	0	1	0	0	0	0	0
4	2	0	3	0	1	0	0	0	0
5	0	5	0	4	0	1	0	0	0
6	5	0	9	0	5	0	1	0	0
7	0	14	0	14	0	6	0	1	0
8	14	0	28	0	20	0	7	0	1

Figure 3: The Jordan block structure of $\alpha^{\otimes k}$, α a Jordan block of size 2

Each Jordan block of size $n \geq 2$ at level k contributes a Jordan block of size $n+1$ and $n-1$ at level $k+1$, that is

$$c_{n,2,k} = c_{n-1,2,k-1} + c_{n+1,2,k-1} \tag{6}$$

If we continue to expand this recurrence we see that $c_{n,2,k} = c_{n-2,2,k-1} + 2c_{n,2,k-2} + c_{n+2,2,k-2}$. Continuing further we see that this expansion appears binomial provided that $n - l > 0$. However, examining the case $c_{1,2,k} = c_{0,2,k-1} + c_{2,2,k-1}$ we see that we must make some sort of correction for $n < 2$. More explicitly we need that $c_{1,2,k} = c_{0,2,k-1} + c_{2,2,k-1} - c_{0,2,k-1}$. To achieve this allow $c_{n,j,k} \neq 0$ for $n < 0$. Then noticing that at level 1 we have a Jordan block of size 2, and at level 3 we must have $c_{0,2,3} = 0$, (6) implies that $c_{-1,2,2} = -1$. Now, again using (6) we obtain figure 4.

$k \diagdown n$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	
2	0	0	0	0	0	-1	0	-1	0	1	0	1	0	0	0	0	0	
3	0	0	0	0	-1	0	-2	0	0	0	2	0	1	0	0	0	0	
4	0	0	0	-1	0	-3	0	-2	0	2	0	3	0	1	0	0	0	
5	0	0	-1	0	-4	0	-5	0	0	0	5	0	4	0	1	0	0	
6	0	-1	0	-5	0	-9	0	-5	0	5	0	9	0	5	0	1	0	
7	-1	0	-6	0	-14	0	-14	0	0	0	14	0	14	0	6	0	1	

Figure 4: The Jordan block structure of $\alpha^{\otimes k}$, α a Jordan block of size 2, allowing for Jordan blocks of size ≤ 0

For $j = 2$ we have the initial conditions that $c_{-1,2,0} = -1$ and $c_{1,2,0} = 1$. Thus, we have that $c_{n,2,k} = [x^n]((x - x^{-1})(x^{-1} + x)^k)$.

Next, consider the case $j \neq 2$. Then we start with one Jordan block of size j. Tensoring this with another Jordan block of size j, again by Li[1], we obtain Jordan blocks of size $2j-1, 2j-3, \cdots, 3, 1$. In general if we have a Jordan block of size $n \geq j$ we produce Jordan blocks of size $n+j-1, n+j-3, \cdots, n-(j-3), n-(j-1)$. Now, if we put this as a recurrence ignoring for now the problems occurring when $n < j$ we have,

$$c_{n,j,k} = \sum_{l=0}^{j} c_{n+j-1-2l,j,k-1} \tag{7}$$

If we continue to expand this recurrence as before we get the initial condition that, $c_{j,j,1} = -c_{-j,j,1} = 1$. Note, that $\forall j \; c_{1,j,0} = -c_{-1,j,0} = 1$. This implies that,

$$\begin{aligned}
c_{n,j,k} &= [x^n]\left(x\left(x^{-j+1} + x^{-j+3} + \cdots + x^{j-3} + x^{j-1}\right)^k - \frac{1}{x}\left(x^{-j+1} + x^{-j+3} + \cdots + x^{j-3} + x^{j-1}\right)^k \right) \\
&= [x^n]\left(\frac{x^2-1}{x}\left(x^{-j+1} + x^{-j+3} + \cdots + x^{j-3} + x^{j-1}\right)^k \right) \\
&= [x^{n+1}]\left((x^2-1)\left(x^{-j+1} + x^{-j+3} + \cdots + x^{j-3} + x^{j-1}\right)^k \right) \\
&= [x^{n+1}]\left((x^2-1)\left(x^{-(j-1)}\left(1 + x^2 + x^4 + \cdots + x^{2(j-1)}\right)\right)^k \right) \\
&= -[x^{n+1+(j-1)k}]\left(\frac{(1-x^{2j})^k}{(1-x^2)^{k-1}} \right)
\end{aligned}$$

Now, to simplify things a bit examine the sequence defined by the denominator. Let,

$$\begin{aligned}
b_n &= [x^n]\frac{1}{(1-x^2)^{k-1}} \\
&= [x^n](1-x^2)^{-(k-1)} \\
&= [x^n]\sum_{n=0}^{\infty}(-1)^n\binom{-(k-1)}{n}x^{2n} \\
&= \begin{cases} (-1)^{\frac{n}{2}}\binom{-(k-1)}{\frac{n}{2}} & if \quad 2|n \\ 0 & otherwise \end{cases} \\
&= \begin{cases} \binom{k+\frac{n}{2}-2}{\frac{n}{2}} & if \quad 2|n \\ 0 & otherwise \end{cases}
\end{aligned}$$

Now, examining the numerator let,

$$\begin{aligned}
a_n &= [x^n](1-x^{2j})^k \\
&= [x^n]\sum_{n=0}^{\infty}(-1)^n\binom{k}{n}x^{2jn} \\
&= \begin{cases} (-1)^{\frac{n}{2j}}\binom{k}{\frac{n}{2j}} & if \quad 2j|n \\ 0 & otherwise \end{cases}
\end{aligned}$$

Next, to obtain $[x^n]\left(\frac{(1-x^{2j})^k}{(1-x^2)^{k-1}}\right)$, we must convolve the sequences a_n and b_n.

$$\begin{aligned}
Q(n,j,k) &= a_n * b_n \\
&= \sum_{l=0}^{n} a_l b_{n-l} \\
&= \begin{cases} \displaystyle\sum_{l}^{\lfloor n/(2j)\rfloor} (-1)^l \binom{k}{l}\binom{k+\frac{n-2jl}{2}-2}{\frac{n-2jl}{2}} & if \quad 2|n \\ 0 & otherwise \end{cases}
\end{aligned}$$

Thus, $c_{n,j,k} = -Q(n+1+(j-1)k, j, k)$. Note that since $Q(n,j,k)$ is non-zero only for even n, if j is odd then we only expect to see odd Jordan block sizes in the recursive tensor. Conversely, if j is even we expect to see odd Jordan block sizes at odd levels and even Jordan block sizes at even levels.

6 A Basis of Eigenvectors for the Tensor of Two Jordan Blocks

Theorem 6.1 *Let $J_n(\alpha)$ and $J_n(\beta)$ be given and let,*

$$A_n := \begin{pmatrix} 0 & \alpha & \alpha & \dots & \alpha \\ 0 & 0 & \alpha & \dots & \alpha \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \alpha \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$B_n := \begin{pmatrix} (-\beta)^{n-1} & 0 & 0 & \dots & 0 \\ 0 & (-\beta)^{n-2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (-\beta)^1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

for integers n and m let $n \leq m$. Let $Z_{i,j}$ denote an $i \times j$ matrix of zeros

$$V_{m,n} := \begin{pmatrix} B_n A_n^0 B_n^{-1} \\ B_n A_n^1 B_n^{-1} \\ B_n A_n^2 B_n^{-1} \\ \vdots \\ B_n A_n^{n-1} B_n^{-1} \\ Z_{mn-n^2,n} \end{pmatrix} \qquad V_{n,m} := \begin{pmatrix} B_n A_n^0 B_n^{-1} \\ Z_{m-n,n} \\ B_n A_n^1 B_n^{-1} \\ Z_{m-n,n} \\ B_n A_n^2 B_n^{-1} \\ \vdots \\ Z_{m-n,n} \\ B_n A_n^{n-1} B_n^{-1} \\ Z_{m-n,n} \end{pmatrix}$$

Then the columns of $V_{m,n}$ and $V_{n,m}$ are a basis, in reduced row eclon form, for the eigenspace of $J_m(\alpha) \otimes J_n(\beta)$ and $J_n(\alpha) \otimes J_m(\beta)$ respectively. Where $\otimes$ denotes the Kronecker tensor of two matrices.

Proof
Let $[a_{i,j}^{(k)}]$ be the ij^{th} element of A_n^k. Then it follows from the definition of A_n^k that

$$a_{i,j}^{(0)} = [x^{j-i}]1 \quad \text{and} \quad a_{i,j}^{(1)} = \alpha[x^{j-i-1}]\frac{1}{1-x}$$

so $\exists k \in \mathbf{Z}$ such that

$$a_{i,j}^{(l)} = \alpha^l [x^{j-i-l}](\frac{1}{1-x})^l \quad \forall l, 1 \geq l \leq k$$

then

$$a_{i,j}^{(k+1)} = \sum_{h=1}^{n} (\alpha^k [x^{j-i-k}](\frac{1}{1-x})^k)(\alpha [x^{j-i-1}]\frac{1}{1-x})$$

which is equivalent to

$$a_{i,j}^{(k+1)} = \sum_{h=1}^{\infty} (\alpha^k [x^{j-i-k}](\frac{1}{1-x})^k)(\alpha [x^{j-i-1}]\frac{1}{1-x})$$

since $i \leq n$. So,

$$a_{i,j}^{(k+1)} = \alpha^{k+1} [x^{j-i-(k+1)}](\frac{1}{1-x})^{k+1}$$

It follows that $B_n A_n^k B_n^{-1} = [\gamma_{i,j}^k]$ such that

$$\gamma_{i,j}^k = \alpha^k (-\beta)^{i-j} [x^{j-i-k}](\frac{1}{1-x})^k$$

Let e_r denote the rth element in the standard basis for a vector space of dimension mn and let w_j denote the jth column of $V_{m,n}$ then $e_{nm} \cdot ((J_m(\alpha) \otimes J_n(\beta))w_j) = \alpha\beta e_{nm} \cdot w_j$

$$\begin{aligned} e_{n(m-1)+i} \cdot ((J_m(\alpha) \otimes J_n(\beta))w_j) &= \alpha\beta e_{n(m-1)+i} \cdot w_j + \alpha e_{n(m-1)+i+1} \cdot w_j && (8) \\ &= \alpha\beta e_{n(m-1)+i} \cdot w_j && (9) \end{aligned}$$

since $e_{nk+r} \cdot w_j = 0 \forall r \geq m-k$

$$\begin{aligned} e_{nk} \cdot ((J_m(\alpha) \otimes J_n(\beta))w_j) &= \alpha\beta e_{nk} \cdot w_j + \beta e_{n(k+1)} \cdot w_j && (10) \\ &= \alpha\beta e_{nk} \cdot w_j && (11) \end{aligned}$$

since $e_{ns} \cdot w_j = 0 \quad \forall \quad s \geq 1$
Note that

$$[x^s](\frac{1}{1-x})^t = \binom{s+t-1}{t}$$

$$
\begin{aligned}
e_{nk+i}\cdot((J_m(\alpha)\otimes J_n(\beta))w_j) &= \alpha\beta e_{nk+i}\cdot w_j + \alpha e_{nk+i+1}\cdot + \ldots \\
&\qquad \ldots w_j \beta e_{n(k+1)+i}\cdot w_j + e_{n(k+1)+i+1}\cdot w_j \\
&= \alpha\beta e_{nk+i}\cdot w_j + \alpha\gamma^{(k)}_{i+1,j} + \beta\gamma^{(k+1)}_{i,j} + \gamma^{(k+1)}_{i+1,j} \\
&= \alpha\beta e_{nk+i}\cdot w_j + \alpha\alpha^k(-\beta)^{i+1-j}[x^{j-(i+1)-k}](\frac{1}{1-x})^k + \ldots \\
&\qquad \ldots \beta\alpha^{k+1}(-\beta)^{i-j}[x^{j-i-(k+1)}](\frac{1}{1-x})^{k+1} + \ldots \\
&\qquad \ldots \alpha^{k+1}(-\beta)^{i+1-j}[x^{j-(i+1)-(k+1)}](\frac{1}{1-x})^{k+1} \\
&= \alpha\beta e_{nk+i}\cdot w_j + \alpha^{k+1}(-\beta)^{i+1-j}([x^{j-i-k-1}](\frac{1}{1-x})^k - \ldots \\
&\qquad \ldots [x^{j-i-k-1}](\frac{1}{1-x})^{k+1} + [x^{j-i-k-2}](\frac{1}{1-x})^{k+1}) \\
&= \alpha\beta e_{nk+i}\cdot w_j + \alpha^{k+1}(-\beta)^{i+1-j}\left(\binom{j-i-2}{k} - \binom{j-i-1}{k+1} - \binom{j-i-2}{k+1}\right) \\
&= \alpha\beta e_{nk+i}\cdot w_j
\end{aligned}
$$

so

$$(J_m(\alpha)\otimes J_n(\beta))w_j = \alpha\beta w_j \forall j$$

The columns of V_m, n are in reduced row echlon form and are therefore linearly independent. Also, they are all eigenvectors with span of dimension $\min(m,n)$. It follows that they form a basis for the eigenspace of $(J_m(\alpha)\otimes J_n(\beta))$.

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