CALCULATING THE ℓ-REGULAR PARTITION FUNCTION

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ABSTRACT. For positive integer n, an ℓ-regular partition of n is a partition of n in which no part is divisible by ℓ. The ℓ-regular partition function $b_\ell(n)$ counts the number of ℓ-regular partitions of n. This paper examines the calculation of $b_\ell(n)$, and gives data evidence for some conjectures about $b_\ell(n)$.

1. INTRODUCTION

A *partition* of a positive integer n is a non-increasing sequence that sums to n, and a *partition function* counts the number of partitions of n. In this paper, we will examine the partition function $b_\ell(n)$, the number of partitions of n into parts not divisible by ℓ.

As stated above, a *partition* of a positive integer n is a non-increasing sequence of positive integers that sums to n For example, the possible partitions of 5 are:

$$
\begin{aligned}
&5\\
&4+1\\
&3+2\\
&3+1+1\\
&2+2+1\\
&2+1+1+1\\
&1+1+1+1+1
\end{aligned}
$$

The (unrestricted) *partition function* $p(n)$ counts the number of partitions of n, so for this example, $p(5) = 7$.

Similarly, we can define other partition functions by limiting what parts are allowable. In particular, we can specify that no part may be divisible by a number ℓ. For $\ell > 1$, an *ℓ-regular partition* of a positive integer n is a partition in which none of the

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parts is divisible by ℓ. For example, the 3-regular partitions of 5 are:

$$\begin{aligned}&5\\&4+1\\&2+2+1\\&2+1+1+1\\&1+1+1+1+1\end{aligned}$$

The *ℓ-regular partition function* $b_\ell(n)$ counts the number of ℓ-regular partitions of a positive integer n [4]. From the above example, we can see that $b_3(5) = 5$.

2. Generating Functions for $b_\ell(n)$

One of the most useful tools in studying partition functions is generating functions. A *generating function* for a sequence $\{a_n\}$ is a function f such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

For example, the sequence $\{i\}_{i=1}^{\infty}$ has generating sequence
$f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$

The partition function describes a sequence $\{p(n)\}_{n=0}^{\infty}$, with $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$, $p(7) = 15$, $p(8) = 22$, $\ldots$, so $p(n)$ can be written as a generating function as well. Let $P(x)$ be the generating function for $\{p(n)\}_{n=0}^{\infty}$, so

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots \tag{2}$$

We can write this sum more succinctly as in infinite product by considering the partition function combinatorially. For example, for the partitions of n into distinct parts, we have the generating function

$$f(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = \prod_{n=1}^{\infty}(1+x^n) \tag{3}$$

since the coefficient of x^n gives the number of ways to write n as the sum of distinct parts. Similarly, we can write

$$P(n) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$
$$= \prod_{n=1}^{\infty}(1 + x^n + x^{2n} + x^{3n} + \cdots) \tag{4}$$

where the first term counts the ways to have 0 ones, 1 one, 2 ones, etc., the second term counts the ways to have 0 twos, 1 two, 2 twos, and so on, so that the coefficient of x^n gives the partition function $p(n)$. Then, given the identity

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + x^{4n} + \cdots \tag{5}$$

this becomes

$$P(n) = \prod_{n=1}^{\infty}(1 - x^n)^{-1} \tag{6}$$

We can simplify this further by utilizing Euler's Pentagonal Number Theorem[2]:

$$\prod_{n=1}^{\infty}(1 - x^n) = 1 + \sum_{j=1}^{\infty}(-1)^j\left(x^{\frac{3j^2+j}{2}} + x^{\frac{3j^2-j}{2}}\right) = \sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{3j^2-j}{2}} \tag{7}$$

This gives the identity

$$\left\{\sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{3j^2-j}{2}}\right\}\sum_{n=0}^{\infty}p(n)x^n = 1$$
$$\{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots\}\sum_{n=0}^{\infty}p(n)x^n = 1 \tag{8}$$

Solving for the coefficient of x^n then gives Euler's identity[2]

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots$$
$$= \sum_{j\in\mathbb{Z}}(-1)^{j+1}p\left(n - \frac{3j^2+j}{2}\right) + \sum_{j\in\mathbb{Z}}(-1)^{j+1}p\left(n - \frac{3j^2-j}{2}\right) \tag{9}$$

Then using this recurrence, we can calculate $p(n)$ from its initial values. This gives us a reasonable way to calculate larger values of the partition function.

We can define similar generating functions for more general partition functions. For example, we may restrict the parts to a specific set, say H, and require that every partition of n be composed of elements of H. Let $p(H,n)$ be the partition function defined in this way. Using the same argument as for the unrestricted partition function, this gives the following generating function[1]:

$$\sum_{i=0}^{\infty} p(H,i)x^i = \prod_{n\in H}(1-x^n)^{-1} \tag{10}$$

From this equation, it is relatively simple to define the infinite product generating function for the ℓ-regular partition function. If we let $H = \{a \in \mathbb{N} : \ell \nmid a\}$, then the partition function $p(H,n) = b_\ell(n)$. The generating function for $b_\ell(n)$ then follows from (10). For example, if $\ell = 3$, $H = \{1,2,4,5,7,8,\ldots\}$, and so $b_3(n)$ would have generating function

$$\begin{aligned}\sum_{n=0}^{\infty} b_3(n)x^n &= \frac{(1-x^3)(1-x^6)(1-x^9)(1-x^{12})(1-x^{15})\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots} \\ &= \frac{\prod_{i=1}^{\infty}(1-x^i)^{-1}}{\prod_{j=1}^{\infty}(1-x^{3j})^{-1}} = \frac{P(x)}{P(x^3)}\end{aligned} \tag{11}$$

where $P(x) = \prod_{n=1}^{\infty}(1-x^n)^{-1}$ is the generating function for the unrestricted partition function. This derivation works for every choice of ℓ, so in general, $b_\ell(n)$ has generating function:

$$\sum_{n=0}^{\infty} b_\ell(n)x^n = \frac{P(x)}{P(x^\ell)} \tag{12}$$

Using (12) and the Pentagonal Number Theorem (7), we can construct a recurrence similar to the Euler recurrence for $p(n)$.

$$f(x) = \frac{P(x)}{P(x^\ell)} \implies \left(\sum_{j\neq 0}(-1)^j x^{\frac{3j^2+j}{2}}\right)\left(\sum_{n=0}^{\infty} b_\ell(n)x^n\right) = \sum_{j\neq 0}(-1)^j x^{\ell\frac{3j^2+j}{2}} \tag{13}$$

Then, solving for the coefficient of x^n, we get:

$$\sum_{\frac{3j^2+j}{2}\leq n} (-1)^j b_\ell\left(n - \frac{3j^2+j}{2}\right) = \begin{cases} (-1)^k & \text{if } n = \ell\left(\frac{3k^2+k}{2}\right) \\ 0 & \text{otherwise}\end{cases} \tag{14}$$

Then if we define a correction factor $\chi_\ell(n)$, with

$$\chi_\ell(n) = \begin{cases} (-1)^k & \text{if } n = \ell\left(\frac{3k^2+k}{2}\right) \\ 0 & \text{otherwise}\end{cases} \tag{15}$$

we can solve for $b_\ell(n)$ to get the recurrence:

$$b_\ell(n) = \sum_{\frac{3j^2+j}{2} \le n-1} (-1)^{j+1} b_\ell\left(n - \frac{3j^2+j}{2}\right) + \chi_\ell(n) \tag{16}$$

This recurrence is identical to the Euler recurrence for the unrestricted partition function $p(n)$, only with the addition of the $\chi_\ell(n)$ term. Calculating $b_\ell(n)$ is then the same as calculating $p(n)$ with this extra term.

3. A Natural Algorithm for Calculating $b_\ell(n)$

4. Data

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