BASES OF SPACES OF MODULAR FORMS WITH WEIGHT $\frac{3}{2}$

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Abstract

The aim of this research is to construct a basis for the space of modular forms with weight $\frac{3}{2}$. Building on the work of Serre and Stark (see [7]), a modular form subspace is initially constructed with certain ternary quadratic forms. These ternary forms result in theta series that are modular forms with weight $\frac{3}{2}$. At various levels N, the dimension of these subspaces are compared with the dimension of the whole modular form space. If necessary, different types of modular forms with weight $\frac{3}{2}$ are added to the subspace. Once the whole space is constructed for numerous levels, a conjecture for the basis of the space is formed. [1]

1 Introduction

In the work of Serre and Stark (see [7]) on modular forms of weight $\frac{1}{2}$, it is suggested that dimension formulas for modular forms of weight $\frac{3}{2}$ can be calculated. The method we use to compute the dimension formulas in [7] was to construct a basis of certain theta series with weight $\frac{1}{2}$. Clearly, once this basis was known, the dimension of the space could be calculated. Although, formulas for the dimension of the space of modular forms with weight $\frac{3}{2}$ are known, it is the goal of our paper to produce 'nicer' bases.

An intermediate goal is to explore the dimension of the space consisting of special theta series with weight $\frac{3}{2}$. Note this is the subspace referred to for the remainder of the paper. Also note that the space $M_{\frac{3}{2}}(N, \chi_0)$ where χ_0 is the trivial character is analyzed in this paper. Will this subspace be the whole space of weight $\frac{3}{2}$ modular forms? If so, is there a nice formula to compute the dimension? Otherwise, what other forms of weight $\frac{3}{2}$ will need to be added to obtain the whole space?

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Nonetheless, our research was conducted during an eight week Research Experience for Undergraduate program. Due to the length of the program, only the intermediate goal mentioned above was completed, that is the exploration of the subspace of modular forms with weight $\frac{3}{2}$ constructed from certain ternary quadratic forms.

First, modular forms, ternary quadratic forms, and the notions of 'lifting' and 'twisting' modular forms are discussed. With this information, a detailed algorithm for the construction of the theta series subspace is given. A program to implement the algorithm in MAPLE mathematical software is also provided in the Appendix. Furthermore, the results of our research are presented in this paper and further progress will be released at a later date.

2 Modular Forms

The following is a brief discussion on modular forms which was derived from [4].

Let the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with c ≡ 0 mod(N) be denoted by $\Gamma_0(N)$.

Definition 2.1. Let k be an integer, N, a natural number and χ be a Dirichlet character modulo N. Denote the upper half complex plane by $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$. A *modular form of weight k, level N and character* χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

- $f(\frac{a\tau+b}{c\tau+d}) = \chi(d)(c\tau + d)^k f(\tau)$ for all $\tau \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$
- f is holomorphic at all cusps of $\mathbb{H}/\Gamma_0(N)$

The space of such functions is denoted $M_k(N, \chi)$. If f vanishes at all cusps of $\mathbb{H}/\Gamma_0(N)$ then f is called a *cusp form*. The subspace of cusp forms is denoted $S_k(N, \chi)$.

Example 2.2. (See [5] for more details.) Let k be an even integer greater than 2. For $z \in \mathbb{H}$, define

$$G_k(z) := \sum_{\substack{m,n \\ m\neq 0, n\neq 0}} \frac{1}{(mz + n)^k}.$$

$G_k(z)$ is called an *Eisenstein series* and is a modular form of weight $2k$.

For the purpose of our research, a similar definition for modular forms of half-integral weight is defined.

Definition 2.3. Let k be an odd integer, N, an integer divisible by 4 and χ be a Dirichlet character modulo N. Denote the upper half complex plane by $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$. A *modular form of weight $\frac{k}{2}$, level N and character* χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

- $f(\frac{a\tau+b}{c\tau+d}) = \begin{cases} \chi(d)\chi_c(d)\epsilon_d^{-k}(\sqrt{c\tau+d})^k f(\tau), & \text{if } c \neq 0 \\ \\ \chi(d)f(\tau), & \text{otherwise} \end{cases}$

  for all $\tau \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

  where $\epsilon_d = \begin{cases} 1 & \text{if d} \equiv 1 \pmod 4 \\ i & \text{if d} \equiv 3 \pmod 4 \end{cases}$

- f is holomorphic at all cusps of $\mathbb{H}/\Gamma_0(N)$

The space of the functions is denoted $M_{\frac{k}{2}}(N,\chi)$ and if f vanishes at all cusps of $\mathbb{H}/\Gamma_0(N)$ then f is called a *cusp form*. The subspace of cusp forms is denoted $S_{\frac{k}{2}}(N,\chi)$.

3 Ternary Quadratic Forms

One way of building modular forms with weight $\frac{3}{2}$ is to use ternary quadratic forms (see [4]). Let Q be the *ternary quadratic form*:

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$$

with a, b, c, r, s, t $\in \mathbb{Z}$. Furthermore, we will only utilize *positive definite ternary quadratic forms*, that is $Q(x,y,z)$ satisfying:

- $Q(x,y,z) \geq 0$ for all $x,y,z \in \mathbb{R}$
- $Q(x,y,z) = 0$ if and only if $x = y = z = 0$.

We will also restrict our attention to the forms that are reduced.

Definition 3.1. Given a ternary quadratic form $Q(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$, Q *is reduced* if all of the following conditions hold:

- $a \leq b \leq c$,
- r, s and t are all positive or all non-positive,
- $|t| \leq a$, $|s| \leq a$ and $|r| \leq b$,
- $a + b + r + s + t \geq 0$,
- If $a + b + r + s + t = 0$, then $2a + 2s + t \leq 0$,
- If $a = -t$, then $s = 0$; if $a = -s$, then $t = 0$; if $b = -r$, then $t = 0$,
- If $a = b$, then $|r| \leq |s|$; if $b = c$, then $|s| \leq |t|$,
- If $a = t$, then $s \leq 2r$; if $a = s$, then $t \leq 2r$; and if $b = r$, then $t \leq 2s$.

Proposition 3.2. *Consider the following theta series*

$$\Theta_Q(\tau) = \sum_{x,y,z\in\mathbb{Z}} q^{Q(x,y,z)}$$

where Q is a reduced positive definite ternary quadratic form and $q = e^{2\pi i\tau}$. Furthermore, $\Theta_Q(\tau) \in M_{\frac{3}{2}}(N_Q, \chi_{d_Q})$.

Proof.We will only prove the weight of $\Theta_Q(\tau)$ here. See [5] for more information. From [7], the space of the Jacobi-Theta series,

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} \text{ where } q = e^{2\pi i\tau}$$

is a space of modular forms with weight $\frac{1}{2}$. The theta series, $\Theta_Q(\tau)$ described above can be constructed with Jacobi-Theta series:

$$\begin{aligned}
\Theta_Q(\tau) &= \sum_{n\in\mathbb{Z}} q^{Q(x,y,z)} \\
&= \sum_{n\in\mathbb{Z}} q^{kx^2+ly^2+mz^2} \\
&= \sum_{k\in\mathbb{Z}} q^{kx^2} \cdot \sum_{l\in\mathbb{Z}} q^{lx^2} \cdot \sum_{m\in\mathbb{Z}} q^{mx^2} \\
&= \Theta(k\tau)\cdot\Theta(l\tau)\cdot\Theta(m\tau)
\end{aligned}$$

where $kx^2 + ly^2 + mz^2$ is derived from Q by completing by square so k, l and $m \in \mathbb{Q}$.

The multiplication of the Jacobi-Theta series does not alter the fact that the product (in this case $\Theta_Q(\tau)$) is a modular form. Furthermore, since weights are additive, $\Theta_Q(\tau)$ has weight $\frac{3}{2}$. □

4 Lifting

Since an intermediate goal of this paper is to build a subspace of the modular form space with the previously described theta series, we implement the following theorem to include more forms in the subspace:

Theorem 4.1. *From [5], if $d_1 d_2 = N$ and $f \in M_k(d_1)$, then we have $f \in M_k(N)$ and also $g(z) := f(d_2 z) \in M_k(N)$.*

Proof.See [5]. □

This method of constructing new modular forms is referred to as 'lifting'.

5 Twisting

Given a level N, we also construct modular forms in $M_{\frac{3}{2}}(N)$ by 'twisting' forms from space of lower level into $M_{\frac{3}{2}}(N)$.

Along with level N and weight k, a modular form is characterized by a Dirichlet character, χ modulo N.

Definition 5.1. (adapted from [1]) Let t be a positive integer and G be the group of reduced residue classes modulo t or the unit group $U(\mathbb{Z}/t\mathbb{Z})$. A *character of* G, $\tilde{\chi}$ is defined as the homomorphism

$$\tilde{\chi} : G \to \mathbb{C} \text{ where } \tilde{\chi}(m) = an\ order(m)^{th}\ root\ of\ unity.$$

For each $\tilde{\chi}$, define the function χ as follows

$$\begin{aligned} \chi(m) = \tilde{\chi}(m) \quad if \quad (m,t) = 1 \\ \chi(m) = 0 \quad if \quad (m,t) \neq 1 \end{aligned}$$

χ is called the *Dirichlet character modulo t.* Note if $g \in G$ and $ord(g) = j$, then $(\chi(g))^j = \chi(g^j) = \chi(1_G) = 1$. Thus, $\chi(g)$ is an j^{th} root of unity.

Furthermore, χ is described by its conductor t.

Definition 5.2. (adapted from [1]) Let χ be a Dirichlet character modulo t and s be any positive divisor of t. s is called an *induced modulus* for χ if $\chi(m)$=1 whenever (m,t)=1 and $m \equiv 1 \pmod{s}$. The smallest induced modulus s for χ is called the *conductor* of χ.

Denote the conductor t for the remainder of the paper. Later in the section on twisting, the notion of primitive characters is discussed.

Definition 5.3. (adapted from [1]) A character χ is said to be *primitive* mod t if χ has no induced modulus $s < t$. This is the same as to say that for any positive divisor $s < t$, there exists an integer m such that (m,t)=1, $m \equiv 1$ (mod s) and $\chi(m) \neq 1$.

Example 5.4. Given level $N = 20$, determine the primitive characters mod 20:

The unit group G is the set {1,3,7,9,11,13,17,19}. Now find the characters of G, $\tilde{\chi} : G \to \mathbb{C}$. To do this, the cyclic decomposition of G must be determined, which is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in this case. Furthermore, generators of each group must be noted. Considering the orders of the elements of G, 3 and 19 are selected as generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively. According to the definition of Dirichlet character, $\chi(3) = \{1,-1,i, \text{ or } -i\}$ and $\chi(19) = \{1 \text{ or } -1\}$. Now there are eight possibilities for Dirichlet characters modulo 20.

Since χ is an homomorphism $\chi(3)$ and $\chi(19)$ determine where the rest of the elements of G get mapped to. For instance, if $\chi(3) = i$ and $\chi(19) = -1$, then $\chi(7) = \chi(3^3 19^0) = (\chi(3))^3(\chi(19))^0 = i^3(-1)^0 = -i$.

Using the following information,

$$1 = 3^0 19^0, \qquad 3 = 3^1 19^0, \qquad 7 = 3^3 19^0, \qquad 9 = 3^2 19^0$$
$$11 = 3^2 19^1, \qquad 13 = 3^3 19^1, \qquad 17 = 3^1 19^1, \qquad 19 = 3^0 19^1$$

the following table is completed:

	$\chi(3)$	$\chi(19)$	$\chi(1)$	$\chi(3)$	$\chi(7)$	$\chi(9)$	$\chi(11)$	$\chi(13)$	$\chi(17)$	$\chi(19)$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	-1	-1	-1	-1
χ_3	-1	1	1	-1	-1	1	1	-1	-1	1
χ_4	-1	-1	1	-1	-1	1	-1	1	1	-1
χ_5	i	1	1	i	$-i$	-1	-1	$-i$	i	1
χ_6	i	-1	1	i	$-i$	-1	1	i	$-i$	-1
χ_7	$-i$	1	1	$-i$	i	-1	-1	i	$-i$	1
χ_8	$-i$	-1	1	$-i$	i	-1	1	$-i$	i	-1

Note that 1 is an induced modulus for χ_1 and 10 is an induced modulus for χ_3 and χ_8 so these are not primitive characters modulo 20. Moreover, the remainder of the Dirichlet characters, χ_2, χ_4, χ_5, χ_6 and χ_7 satisfy the conditions to be primitive characters.

Theorem 5.5. *From [5], if $f = \sum_{n\geq 0} a_n q^n \in M_k(M, \Psi)$ and χ is a primitive character modulo t, then $g(z) := \sum_{n\geq 0} a_n \chi(n) q^n \in M_k(Mt^2, \Psi\chi^2)$.*

Proof. See [5]. $\square$

Note that if no character is listed, then the trivial character is assumed.

6 Computation

Using the Theorems 4.1 and 5.5, the following subsections describe an algorithm to determine the dimension of a subspace of modular forms for a given level N with trivial character and weight $\frac{3}{2}$. We will construct forms directly and by lifting and twisting.

$$\boxed{M_{\frac{3}{2}}(D, \chi_0)} \overrightarrow{lifting} \boxed{M_{\frac{3}{2}}(N, \chi_0)} \overleftarrow{twisting\ by\ \chi} \boxed{M_{\frac{3}{2}}(\tfrac{N}{t^2}, \chi^{-2})}$$

where D is a divisor of the level N and χ_0 is the trivial character for the purposes of this paper.

6.1 Building forms in $M_{\frac{3}{2}}(N)$ directly

For a given level N, we construct all reduced positive definite ternary quadratic forms (ternary forms for short) with Lehman's algorithm (see [6]):

If $Q(x, y, z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ is the desired ternary form, then its discriminant is given by $d = 4abc + rst - ar^2 - bs^2 - ct^2$ and its level is given by $N = 4d/m$ where m is the greatest common divisor of $4bc - r^2$, $4ac - s^2$, $4ab - t^2$, $2st - 4bs$, $2rt - 4bs$ and $2rs - 4ct$. Since Q is positive definite, each of a, b, c, $4bc - r^2$, $4ac - s^2$, $4ab - t^2$ and d is greater than 0.

1. Given N and d, let $m = 4d/N$ and $\mu = 4N/m$. If m and μ are not integers, then there are no ternary forms of this level and discriminant.

2. Let a vary so that:
   $1 \leq a \leq \lfloor \sqrt[3]{\frac{d}{2}} \rfloor$ and if μ is odd, then $a \equiv 0$ or $-\mu (\text{mod } 4)$.

3. For each such a, find g, u and $v \in \mathbb{Z}$ where g=gcd($4a$,m) and $g = 4au + mv$.

4. Let t vary so that:
   $0 \leq t \leq a$, if m is even, then t is even and $g | t^2$.

5. Let b vary so that:
   $b \equiv \frac{ut^2}{g}$ (mod $\frac{m}{g}$), $\max(a, \frac{m}{4a}) \leq b \leq \sqrt{\frac{d}{2a}}$ and
   if μ is odd, then $b \equiv 0$ or $-\mu$ (mod 4).

6. Let s vary so that:
   $0 \leq s \leq a$, if m is even, then s is even and $g | 2st$.

7. Let r vary so that:
   $r \equiv \frac{2stu}{g}$ (mod $\frac{m}{g}$), $|r| \leq b$, if m is even, then r is even and if $st = 0$, then $r \leq 0$.

8. For c:
   Let $c = \frac{d - rst + ar^2 + bs^2}{4ab - t^2}$. If $c \in \mathbb{Z}$, then $Q(x, y, z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ is a candidate for the desired ternary form.

9. Double check to see if the coefficients satisfy the conditions to be a reduced form (see section 3). Also, check if m is indeed the greatest common divisor of $4bc - r^2$, $4ac - s^2$, $4ab - t^2$, $2st - 4bs$, $2rt - 4bs$ and $2rs - 4ct$ and if μ is odd, then $c \equiv 0$ or $-\mu (\text{mod } 4)$.

Since we are only using trivial characters, given N, consider only discriminants, d, such that $d | N^2$.

Example 6.1. Finding reduced positive definite ternary quadratic forms of level $N = 8$ and discriminant $d = 16$:

1. Since $m = 4d/N$ and $\mu = 4N/m$, $m = 8$ and $\mu = 4 \in \mathbb{Z}$.

2. The variable a would vary between 1 and $\lfloor \sqrt[3]{\frac{d}{2}} \rfloor = 2$ so ***a=1 or 2***. Furthermore, μ is even so the second condition on a does not apply. (The $a = 2$ case will not be considered for the remainder of the example. It fails at step 8.)

3. Now $g = \gcd(4, 8) = 4$ so for $4 = 4au + mv$, $u = 1$ and $v = 0$.

4. The variable t would vary such that $0 \leq t \leq 1$ and since m is even, t is even. ***t=0*** is the only candidate which holds since $4|0$.

5. For b, $b \equiv 0(\mod 2)$ so that b is between $\max(1, 2) = 2$ and $\sqrt{8}$. Since μ is even, the last condition on b does not apply so ***b=2***.

6. For s, s is between 0 and 1 and m being even implies the only candidate is $s = 0$. Since $4|0$, ***s=0***.

7. Now since $r \equiv 0(\mod 2)$, $|r| \leq 2$ and m is even so r= -2, 0 or 2. Also, $st = 0$ so ***r= -2 or 0***.

8. Given d, a, t, b, s and r, test if $c = \frac{d - rst + ar^2 + bs^2}{4ab - t^2}$ is an integer. With $r = -2$, $c = \frac{5}{4} \notin \mathbb{Z}$, but with $r = 0$, ***c=2*** $\in \mathbb{Z}$.

9. Now $Q(x, y, z) = x^2 + 2y^2 + 2z^2$ is a potential ternary form. In fact, a, b, c, r, s and t satisfy the conditions for a reduced ternary form and $m = \gcd(8, 8, 8, 0, 0, 0) = 8$. So Q is the desired ternary form.

Now reduced positive definite ternary quadratic forms $Q(x, y, z)$ of a given level N are available. Consider the theta series

$$\Theta_Q(\tau) = \sum_{x,y,z \in \mathbb{Z}} q^{Q(x,y,z)} = \sum_{n \in \mathbb{Z}} r_n q^n$$

where $Q(x, y, z) = n$ and r_n are the Fourier coefficients of the modular form, $\Theta_Q(\tau) \in M_{\frac{3}{2}}(N)$. r_n is also the number of ways x, y and z can represent n in $Q(x, y, z) = n$. Since the coefficients r_n uniquely determine the modular form, it is sufficient to use only the r_n's to describe the space. The following explains how the coefficients are calculated.

CALCULATING THE FOURIER COEFFICIENTS OF A MODULAR FORM GIVEN A TERNARY FORM

Fortunately given the level N, the coefficients of the modular form only need to be calculated up to a certain bound. The bound is given by the following theorem:

Theorem 6.2. *(see [4]) Suppose that f, $g \in M_{k/2}(N, \chi)$ where k is odd and $4|N$. Suppose also that $r_n(f) = r_n(g)$ for $0 \leq n \leq \frac{(k+1)N}{24} \prod_{p|N}(1 + \frac{1}{p})$. Then f=g.*

Therefore, the coefficients $r_0, r_1, \ldots r_n$ where $0 \leq n \leq \frac{N}{6}\prod_{p|N}(1+\frac{1}{p})$ are required to determine modular forms $\in M_k(N)$. The calculation goes as follows

1. Given the level N, calculate the upper bound for the coefficients, $\frac{N}{6}\prod_{p|N}(1+\frac{1}{p})$. Denote it B.
2. Determine the range for which x, y and z can represent n. The range is given for x, y and z are

$$\lfloor -\sqrt{\frac{B}{\alpha}} \rfloor \leq \; x \; \leq \lceil \sqrt{\frac{B}{\alpha}} \rceil, \quad \lfloor -\sqrt{\frac{B}{\beta}} \rfloor \leq \; y \; \leq \lceil \sqrt{\frac{B}{\beta}} \rceil,$$

$$\text{and } \lfloor -\sqrt{\frac{B}{\gamma}} \rfloor \leq \; z \; \leq \lceil \sqrt{\frac{B}{\gamma}} \rceil.$$

   where α, β, and γ are defined as

$$\alpha = -\frac{a(-4bc+r^2)+ct^2+s(-rt+bs)}{4bc-r^2},$$

$$\beta = -\frac{b(-4ac+s^2)+ct^2+r(-st+ar)}{4ac-s^2},$$

$$\gamma = -\frac{c(-4ba+t^2)+ar^2+s(-rt+bs)}{4ba-t^2}$$

3. Now for each n, store the coefficients r_n,

$$\left[\begin{array}{c|c|c|c|c} n & 0 & 1 & \ldots & B \\ \hline r_n & * & * & \ldots & * \end{array}\right]$$

Example 6.3. Continuing with Example 6.1, the Fourier coefficients of the ternary form $Q(x,y,z) = x^2+2y^2+2z^2$ are calculated to distinguish a modular form in $M_{\frac{3}{2}}(8)$.

1. First, the number of coefficients needs to be determined by computing the upper bound B. Since $N = 8$,

$$B \; = \; \frac{8}{6}\prod_{p|8}(1+\frac{1}{p}) \; = \; \frac{8}{6}\cdot(1+\frac{1}{2}) \; = \; 2.$$

   Thus, only r_0, r_1 and r_2 are required.
2. Next, the ranges for x, y and z are needed.

$$\alpha \; = \; - \; \frac{1(-16+0)+0+0}{16-0} \; = \; 1$$

$$\begin{aligned}\beta &= -\frac{2(-8+0)+0+0}{8-0} = 2\\ \gamma &= -\frac{2(-8+0)+0+0}{8-0} = 2\end{aligned}$$

Hence,

$$-2 \le x \le 2, \quad -1 \le y \le 1, \text{ and } -1 \le z \le 1.$$

3. With these defined bounds on x, y and z in $Q(x,y,z) = x^2+2y^2+2z^2 = n$, there is 1 way to represent 0, 2 ways to represent 1 and 4 ways to represent 2. So the modular form is stored in the array:

$$\left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 2 & 4 \end{array}\right]$$

6.2 Lifting Modular Forms into $M_{\frac{3}{2}}(N)$

Lifting modular forms can be executed in two fashions:

1. According to Theorems 4.1 and 6.2, for each D such that $4|D|N$, $M_{\frac{3}{2}}(D) \subseteq M_{\frac{3}{2}}(N)$. Therefore, build forms directly as in section 6.1 for each D such that $4|D|N$.

2. Furthermore by Theorem 4.1, $g(z) := f(\frac{N}{D}z) \in M_{\frac{3}{2}}(N)$ for all D such that $4|D|N$. If $f(z) = \sum_{n\ge 0} r_n q^n$, then

$$f(\frac{N}{D}z) = \sum_{n\ge 0} r_n (q^{\frac{N}{D}})^n = \sum_{n \ge 0} r_{\frac{n\cdot D}{N}} q^n.$$

   Thus, the coefficients of f can be dispersed by a factor of $\frac{N}{D}$ to obtain another form in $M_{\frac{3}{2}}(N)$.

Example 6.4. Lifting modular forms into $M_{\frac{3}{2}}(8)$

1. Modular forms in $M_{\frac{3}{2}}(D)$ are also in $M_{\frac{3}{2}}(8)$ where $D = 4$ or 8. Using the method of producing modular forms described in subsection 6.1, the following is generated:
   $D = 4$ yields

$$\left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 6 & 12 \end{array}\right]$$

   and $D = 8$ yields

$$\left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 2 & 4 \end{array}\right]$$

2. Moreover, the 'dispersement factor' for $D = 4$ is $\frac{N}{D} = 2$ so the following form can be accounted for

$$\left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 0 & 6 \end{array}\right]$$

Since the 'dispersement factor' for $D = 8$ is 1, the form is already constructed using the direct method.

6.3 Twisting Modular Forms into $M_{\frac{3}{2}}(N)$

Along with building forms in $M_{\frac{3}{2}}(N)$ directly and lifting forms from $M_{\frac{3}{2}}(D)$, modular forms in $M_{\frac{3}{2}}(N)$ can be twisted up from a lower space. By theorem 5.2, forms in $M_{\frac{3}{2}}(\frac{N}{t^2}, \chi^{-2})$ can be twisted by the character χ with conductor t into $M_{\frac{3}{2}}(N)$.

To do this, we use the following algorithm:

1. For a given level N, choose s such that $s^2 | \frac{N}{4}$.

2. For each such s, find the cyclic decomposition of the unit group $U(\mathbb{Z}/s\mathbb{Z})$. In other words, find $n_1, n_2, \ldots, n_r$ such that $U(\mathbb{Z}/s\mathbb{Z}) \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$. This can be done by using the theorem:

   Theorem 6.5. *(see [3]). Let $n = 2^a p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l}$ be the prime decomposition of n. Then*

   $$U(\mathbb{Z}/n\mathbb{Z}) \cong U(\mathbb{Z}/2^a\mathbb{Z}) \times U(\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \times \cdots \times U(\mathbb{Z}/p_l^{a_l}\mathbb{Z}).$$

   $U(\mathbb{Z}/p_i^{a_i}\mathbb{Z})$ is a cyclic group of order $p_i^{a_i-1}(p_i - 1)$. $U(\mathbb{Z}/2^a\mathbb{Z})$ is cyclic of order 1 or 2. If $a = 1$ and 2, respectively. If $a \geq 3$, then it is the product of two cyclic groups, one of order 2, the other of order 2^{a-2}.

3. Find generators g_i such that $< g_i >= \mathbb{Z}/n_i\mathbb{Z}$ (from step 2) for all i.

4. Develop a list of Dirichlet characters, $\tilde{\chi} : U((\mathbb{Z}/s\mathbb{Z}) \to \mathbb{C}$ that satisfy $\tilde{\chi}(g_i) = (n_i, 4)^{th} \textit{root of unity}$.

5. Recall $\frac{N}{s^2}$ is the new level of the current space. Now find discriminants d such that $d | \frac{N^2}{t^4}$ and where $\chi_j^{-2}(m) = \left(\frac{d}{m}\right)$ for all $m \in U(\mathbb{Z}/s\mathbb{Z})$.

6. Compute the ternary forms and Fourier coefficients directly with level $\frac{N}{s^2}$ and discriminants d for all d from previous step. This gives modular forms in $M_{\frac{3}{2}}(\frac{N}{s^2}, \chi^{-2})$.

7. For each modular form constructed, multiply r_m by $\tilde{\chi}(m)$ if $(m, s) = 1$. Otherwise $r_m = 0$. This multiplication twists the modular forms by character χ into $M_{\frac{3}{2}}(\frac{N}{s^2} \cdot t^2)$ where t is the conductor of χ.

8. If s equals the conductor t, then the twisted modular forms are in fact in $M_{\frac{3}{2}}(N)$. However, if s does not equal t, then the forms must be lifted into $M_{\frac{3}{2}}(N)$ (see section 4 and previous section on Lifting).

Due to the complexity of the twisting computation, a 'nice' example is not provided in this paper. For an illustration of its implementation, see Appendix.

6.4 Determining the Dimension of the Subspace

Now that all of the Fourier coefficients are obtained, the dimension of the constructed 'ternary form' subspace of modular forms can be calculated as follows:

1. Input all of the Fourier coefficients from the previous step into a matrix

$$\begin{pmatrix} r_{1_0} & r_{1_1} & r_{1_2} & \cdots & r_{1_B} \\ r_{2_0} & r_{2_1} & r_{2_2} & \cdots & r_{2_B} \\ r_{3_0} & r_{3_1} & r_{3_2} & \cdots & r_{3_B} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{t_0} & r_{t_1} & r_{t_2} & \cdots & r_{t_B} \end{pmatrix}$$

   where r_{i_j} is the j^{th} coefficient of the i^{th} modular form.

2. Use Gaussian elimination to find the rank, R, of this matrix. R is the dimension of the subspace with the given level N.

Example 6.6. Using Examples 6.3 and 6.4, the dimension of the subspace of modular forms derived by ternary forms will be computed for level N=8.

1. From building modular forms directly and lifting, the following forms were constructed in $M_{\frac{3}{2}}(8)$:

$$\left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 6 & 12 \end{array}\right], \left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 2 & 4 \end{array}\right], \left[\begin{array}{c|c|c|c} n & 0 & 1 & 2 \\ \hline r_n & 1 & 0 & 6 \end{array}\right]$$

   Putting this coefficients into matrix produces:

$$\begin{pmatrix} 1 & 6 & 12 \\ 1 & 2 & 4 \\ 1 & 0 & 6 \end{pmatrix}$$

2. Gaussian elimination yields

$$\begin{pmatrix} 1 & 6 & 12 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

   which has rank 3. Therefore, the dimension of the subspace at level 8 is 3.

In this example twisting is not necessary because the dimension of the subspace is equal to the dimension of $M_{\frac{3}{2}}(8)$ (see 0* in the figures of the Results section).

7 Results

The data for the levels, $N \leq 200$ is given in the tables. The dimensions for the whole space of modular forms $M_{\frac{3}{2}}(N)$ was provided by [2]. Note that it is assumed in [2] that the dimension exactly at N were calculated. Since our interest is in the dimensions at levels dividing N for lifting and twisting, new formulas were used for the data. Those new dimensions are in the columns denoted "0*" in the following tables.

The following table shows our results of constructing the subspace of modular forms directly without lifting or twisting. In only one case is the whole space obtained which is at level 4. However, in most cases not even a quarter of the whole space is formed. For instance at level 188, merely 6.7% of the space is constructed via this method.

LEVEL	0*	1*	1*/0*	LEVEL	0*	1*	1*/0*
4	1	1	100.0	104	18	4	22.2
8	3	1	33.3	108	25	8	32.0
12	4	2	50.0	112	32	8	25.0
16	7	3	42.9	116	10	1	10.0
20	4	1	25.0	120	48	9	18.8
24	11	4	36.4	124	11	1	9.1
28	5	1	20.0	128	39	14	35.9
32	13	5	38.5	132	25	5	20.0
36	10	5	50.0	136	21	4	19.0
40	12	3	25.0	140	24	5	20.8
44	6	1	16.7	144	67	19	28.4
48	25	7	28.0	148	12	1	8.3
52	6	1	16.7	152	23	3	13.0
56	14	4	28.6	156	27	4	14.8
60	17	5	29.4	160	54	14	25.9
64	23	8	34.8	164	13	1	7.7
68	7	1	14.3	168	56	9	16.1
72	29	10	34.5	172	14	1	7.1
76	8	1	12.5	176	39	11	28.2
80	28	7	25.0	180	47	14	29.8
84	20	5	25.0	184	26	3	11.5
88	17	3	17.6	188	15	1	6.7
92	9	1	11.1	192	85	22	25.9
96	47	13	27.7	196	25	9	36.0
100	16	6	37.5	200	48	12	25.0

Figure 1: 0*= dimension of $M_{\frac{3}{2}}(N)$
1*= dimension of the subspace without lifting or twisting

The data below depicts the dimensions of subspaces at various level N after modular forms are constructed directly and lifted. Similarly, there are only a few cases where the dimension of the subspace is the dimension of the whole space (at levels 4, 8 and 12). The dimension of the subspace at level 188 still relatively low with 20.0% of the space obtained.

LEVEL	0*	2*	2*/0*	LEVEL	0*	2*	2*/0*
4	1	1	100.0	104	18	10	55.6
8	3	3	100.0	108	25	18	72.0
12	4	4	100.0	112	32	21	65.6
16	7	5	71.4	116	10	3	30.0
20	4	3	75.0	120	48	35	72.9
24	11	8	72.7	124	11	3	27.3
28	5	3	60.0	128	39	21	53.8
32	13	8	61.5	132	25	13	52.0
36	10	8	80.0	136	21	10	47.6
40	12	9	75.0	140	24	11	45.8
44	6	3	50.0	144	67	37	55.2
48	25	15	60.0	148	12	3	25.0
52	6	3	50.0	152	23	9	39.1
56	14	10	71.4	156	27	12	44.4
60	17	12	70.6	160	54	34	63.0
64	23	13	56.5	164	13	3	23.1
68	7	3	42.9	168	56	40	71.4
72	29	18	62.1	172	14	3	21.4
76	8	3	37.5	176	39	23	59.0
80	28	19	67.9	180	47	34	72.3
84	20	13	65.0	184	26	9	34.6
88	17	9	52.9	188	15	3	20.0
92	9	3	33.3	192	85	48	56.5
96	47	27	57.4	196	25	12	48.0
100	16	8	50.0	200	48	29	60.4

Figure 2: 0*= dimension of $M_{\frac{3}{2}}(N)$; 2*= dimension after lifting

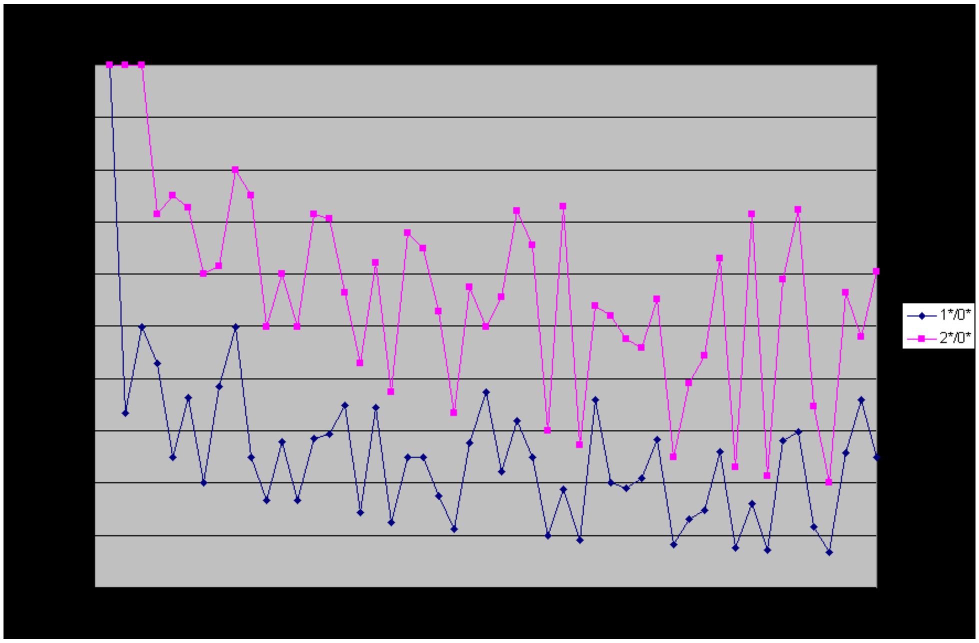

Figure 1: Levels N vs. Methods of Building Subspaces

8 Future Work

Since the whole space was not obtained from the subspace at all levels N, levels $N > 200$ will be investigated. Furthermore, we will also consider other modular forms of weight $\frac{3}{2}$. For instance, we could use Eisenstein series (see Example 2.2). When the whole subspace is achieved through the addition of various forms, bases and dimension formulas will be conjectured.

9 Appendix

The following is an algorithm for MAPLE mathematical software and results for $N \leq 5000$.

```
#  formsandlift computes the dimension, r, (with or without lifting **)
#    of a particular level N
#  all computes the dimensions of all levels up to the level N

formsandlift:=proc(N)
local D,e,p,prod1,i,d,k,l,m,mu,j,ce,fl,divisor,div,ma,var1,i1,L2
,result,mtxval,
M,G,rank,sq,m1,m2,m3,m4,m5:
global a,b,c,r,s,t,u,v,g,R,upbound,q:
mtxval:=[]:
with(numtheory):
with(LinearAlgebra):



# For each D such that divides 4|D|N, calculate the modular forms
# of M(3/2)(D).
# This implements lifting.
# To get the information about a particular level, just go from
# N to N**
for D from 4 to N by 4 do
  if (N mod D=0) then
   q:=N/D:




#To get upper bound for n in r(n), N(k+1)/24*prod(1+1/p),
# where k=3
prod1:= 1:
for e from 1 to N do
   if isprime(e) then
```

```
     if (N mod e = 0) then
       p:=e:
       prod1:= prod1 * (1 + (1/p)):
     fi:
    fi:
od:
upbound:=(N/6)*prod1:



#To get the reduced positive definite ternary forms,
# see section 6.1
d:=0:
divisor:=[]:
div:=tau(D^2):
divisor:=divisors(D^2):
#print('discriminant',divisors(D^2)):


   # only consider the discriminants that are perfect
   # squares which
   # are associated with trivial characters
   for i from 1 to div do
   d:=divisor[i]:
   sq:=sqrt(d):
   if type(sq,integer) then
    #for m and mu, step 1
   m:=(4*d)/D:
   mu:=(4*D)/m:
    if type(m,integer) then
     if type(mu,integer) then
      ma:=floor((d/2)^(1/3)):

    #for a, step 2:
     for a from 1 to ma do
      if mu mod 2=0  or a mod 4=0  or -a mod 4  =(mu)  then
       var1:=4*a:
        euclid(var1,m):

    #for t, step 4:
     for t from 0 to a do
       if( (m mod 2=0  and t mod 2=0 ) or m mod 2=1 ) then
         if t^2 mod g=0  then

    #for b, step 5:
     ce:=ceil(max(a,m/(4*a))):
```

```
   fl:=floor(sqrt(d/(2*a))):
     for k from ce to fl do
       if k mod (m/g) = (u*t^2/g) then
         b:=k:
           if (b >= ce and b <= fl) then
             if ((mu mod 2 =0) or (b mod 4=0) or
                (-b mod 4 =(mu)) )then

#for s, step 6:
 for s from 0 to a do
   if (((m mod 2 =0) and (s mod 2=0 )) or
    (m mod 2 =1)) then
     if (((2*s*t) mod g)=0)  then

#for r, step 7:
 for l from -b to b do:
   if (l mod (m/g) = (2*s*t*u)/g) then
    if (((m mod 2=0) and (l mod 2=0) ) or
     (m mod 2=1))  then
      if (((s=0 or t=0) and  l<= 0) or
        (s <> 0 and t <> 0)) then
        r:=l:

#for c, step 8:
  c:=(d - r*s*t + a*r^2 + b*s^2)/(4*a*b - t^2):
    result:=check(a,b,c,r,s,t):
      if type(c,integer) and  result=1 then
        if mu mod 2=0  or c mod 4=0  or
          -c mod 4  =(mu)  then

 # step 9:
  m1:=gcd(4*b*c-r^2,4*a*c-s^2):
  m2:=gcd(4*a*b-t^2,2*s*t-4*a*r):
  m3:=gcd(2*r*t-4*b*s,2*r*s-4*c*t):
  m4:=gcd(m1,m2):
  m5:=gcd(m3,m4):
    if m=m5 then
     rns(upbound):
      mtxval:=[op(mtxval),mtxval1,mtxval2]:

    fi:
        fi:
      fi:
    fi:
   fi:
  fi:
```

```
     od: (l)

       fi:
      fi:
     od: (s)

        fi:
       fi:
      fi:
     od: (k)

       fi:
      fi:
     od: (t)

      fi:
     od: (a)

        fi:
       fi:
      fi:
     od: (i)


fi:
od:
M:=convert(mtxval,Matrix):
G:=GaussianElimination(M):
R:=Rank(G):
print(R):
end:


#==================================================================================
# to do several levels
all:=proc(q):


for N from 1 to q do
  if N mod 4 = 0 then
   print('N',N):
   formsandlift(N):
  fi:
od:
end:
```

```
#================================================================================
#THE EXTERNAL PROGRAMS
#Step 9 , double checking the 15 conditions
# for a reduced form
check:=proc(a,b,c,r,s,t)
local sum:
if a<=b and b<=c then
if (r>0 and s>0 and t>0 ) or (r<=0 and s<=0 and t<=0) then
if (abs(r)<=b and abs(s)<=a and abs(t)<=a)then
 sum:=a+b+r+s+t:
if (sum>=0) then
if ((sum =0 and (2*a + 2*s + t) <=0) or (sum<>0) )then
if ((a=b and abs(r)<=abs(s)) or a<>b) then
if ((b=c and abs(s)<=abs(t)) or b<>c) then
if ((a=(-t) and s=0) or a<>-t) then
if ((a=(-s) and t=0) or a<>-s) then
if ((b=(-r) and t=0) or b<>-r) then
if ((a=t and s<=2*r) or a<>t)  then
if ((a=s and t<=2*r) or a<>s)  then
if ((b=r and t<=2*s) or b<>r)  then
return 1:
else return 0:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
fi:
end:



# to produce the Fourier coefficients in an array
rns:=proc(upbound)
global a,b,c,r,s,t,L2,q,mtxval1,mtxval2,alpa,betha,
gama,e1,e2,e3,e4,e5,e6,u1:
local j,minu1,maxu1,n,rn,x1,y1,z1,n1,l,i,var,R,k:
mtxval1:=[]:
```

```
mtxval2:=[]:


    #to calculate the Fourier coefficients, r(n)
    u1:=floor(upbound):
    alpa:= -(a*(-4*b*c+r^2)+c*t^2+s*(-r*t+b*s))/(4*b*c-r^2):
    betha:= -(b*(-4*a*c+s^2)+c*t^2+r*(-s*t+a*r))/(4*a*c-s^2):
    gama:= -(c*(-4*b*a+t^2)+a*r^2+s*(-r*t+b*s))/(4*b*a-t^2):
    e1:=floor(-sqrt(upbound/alpa)):
    e2:=ceil(sqrt(upbound/alpa)):
    e3:=floor(-sqrt(upbound/betha)):
    e4:=ceil(sqrt(upbound/betha)):
    e5:=floor(-sqrt(upbound/gama)):
    e6:=ceil(sqrt(upbound/gama)):

    R:=array(1..2,1..u1+1):
    L2:=array(1..2,1..u1+1):
      for n from 0 to u1 do
       rn:=0:
        for x1 from e1 to e2 do
         for y1 from e3 to e4 do
           for z1 from e5 to e6 do
            n1:=a*x1^2+b*y1^2+c*z1^2+r*y1*z1+s*x1*z1+t*x1*y1:
             if (n1=n) then
               rn:=rn + 1:
             fi:
           od:
         od:
       od:
       R[1,n+1]:=n:
       R[2,n+1]:=rn:
       mtxval1:=[op(mtxval1),rn]:
     od:
   for i from 1 to u1+1 do
    L2[1,i]:=i-1:
    L2[2,i]:=0:
   od:
   for j from 1 to u1+1 do
    var:=R[1,j]*q:
     if var=0 then
      L2[2,1]:=R[2,1]:
       elif var<>0 and var <= u1 then
       L2[2,var+1]:=R[2,j]:
     fi:
   od:
  for k from 1 to u1+1 do
```

```
   mtxval2:=[op(mtxval2),L2[2,k]]:
  od:
   # mtxval1 and mtxval2 are the Fourier coefficients
   # to put into the final matrix to be Gaussian eliminated
end:

#Euclidean Algorithm (from Dr. Nigel Byott, University of Exeter)
euclid:=proc(a1,b1)
 local b,q,r,x1,x,y1,y:
 global g,u,v:
 g:=abs(a1):
 b:=abs(b1):
 u:=sign(a1):
 x1:=0:
 v:=0:
 y1:=sign(b1):
 while b<>0 do
 q:=iquo(g,b):
 r:=g-q*b:
 g:=b;b:=r:
 x:=u-x1*q:
 u:=x1:
 x1:=x:
 y:=v-y1*q:
 v:=y1:
 y1:=y:
 od:
end:
```

Here is data for $200 \leq N \leq 1000$:

Level	1*	2*	Level	1*	2*	Level	1*	2*	Level	1*	2*
204	4	12	404	1	3	604	1	3	804	4	12
208	6	21	408	8	37	608	16	42	808	4	10
212	1	3	412	1	3	612	14	35	812	5	11
216	14	43	416	15	39	616	11	45	816	25	82
220	5	11	420	15	53	620	4	10	820	3	9
224	13	37	424	4	10	624	29	80	824	3	9
228	4	12	428	1	3	628	1	3	828	19	42
232	4	10	432	43	81	632	3	9	832	29	74
236	1	3	436	1	3	636	4	12	836	8	14
240	26	67	440	10	40	640	46	108	840	27	160
244	1	3	444	4	12	644	7	13	844	1	3
248	3	9	448	26	63	648	30	92	848	8	23
252	14	37	452	1	3	652	1	3	852	4	12
256	19	35	456	11	38	656	8	23	856	3	9
260	3	9	460	4	10	660	15	54	860	3	9
264	10	39	464	8	23	664	3	9	864	76	160
268	1	3	468	13	34	668	1	3	868	6	12
272	7	22	472	3	9	672	40	146	872	4	10
276	5	13	476	5	11	676	10	13	876	4	12
280	9	40	480	41	135	680	11	39	880	32	100
284	1	3	484	11	14	684	13	34	884	5	11
288	35	65	488	4	10	688	8	21	888	8	37
292	1	3	492	4	12	692	1	3	892	1	3
296	4	10	496	12	25	696	9	38	896	45	113
300	17	42	500	12	24	700	19	43	900	49	108
304	8	21	504	23	104	704	35	78	904	4	10
308	7	13	508	1	3	708	4	12	908	1	3
312	7	36	512	34	53	712	4	10	912	21	84
316	1	3	516	4	12	716	1	3	916	1	3
320	28	62	520	10	38	720	68	175	920	11	39
324	16	34	524	1	3	724	1	3	924	15	59
328	4	10	528	21	88	728	10	43	928	14	43
332	1	3	532	6	12	732	4	12	932	1	3
336	21	76	536	3	9	736	17	52	936	32	110
340	3	9	540	25	78	740	3	9	940	4	10
344	3	9	544	16	42	744	10	37	944	14	27
348	4	12	548	1	3	748	7	13	948	4	12
352	16	46	552	11	40	752	17	30	952	11	44
356	1	3	556	1	3	756	26	83	956	1	3
360	26	92	560	28	91	760	10	40	960	74	243
364	5	11	564	4	12	764	1	3	964	1	3
368	13	26	568	3	9	768	53	140	968	19	49
372	4	12	572	6	12	772	1	3	972	41	75
376	3	9	576	58	117	776	4	10	976	6	21
380	5	11	580	3	9	780	15	48	980	20	51
384	39	83	584	4	10	784	37	81	984	8	37
388	1	3	588	18	51	788	1	3	988	6	12
392	15	42	592	6	21	792	28	113	992	18	51
396	15	38	596	1	3	796	1	3	996	4	12
400	28	55	600	29	118	800	57	108	1000	25	83

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