Expected Value of 4-tuples on Random Cyclic Graphs

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A *cyclic graph* is a graph which is invariant under cyclic automorphism, or, a graph in which any edge drawn forces all other edges of the same distance to be drawn. For example, on a graph of seven vertices with vertex set $V = \{0, 1, \ldots, 6\}$, drawing an edge between 0 and 2 will force an edge between 1 and 3, 2 and 4, 3 and 5, and so on until 6 to 1 has been drawn, meaning every vertex n is connected to both vertices $n + 2$ and $n - 2$, mod 7. A *random cyclic graph* is a cyclic graph in which every independent edge (an edge not forced by already existing edges) is chosen with a probability of one-half. For our purposes, we are assuming N, where $N = |V|$, to be prime.

Because of the nature of cyclic graphs, only $\frac{N-1}{2}$ edges need to be chosen to completely determine the graph. This is clear upon the realization that once one edge of a given distance exists, so do all others. Hence, in order to completely determine the graph, we need only look at whether there are edges between 0 and any vertex from 1 to $\frac{N-1}{2}$. This covers all possible distances between two vertices, and thus it covers all possible edges.

The expected value of the number of complete subgraphs of size four, or 4-tuples, contained in a random cyclic graph on N vertices can be calculated using a sum of the expected value of each type of 4-tuple that could possibly exist, where a type is determined by the number of independent and dependent edges in the 4-tuple. For example, in most cases, it is possible to have 4-tuples with up to three dependent edges. However, on $N = 5$, and only on $N = 5$, it is possible to have a 4-tuple with only two independent edges, and in fact this is the only way to get a 4-tuple. On $N = 7$, it is not possible to have more than three independent edges, and on $N = 11$, there can be no more than five independent edges. Our derivation of the expected value formula was done using $N = 19$ to ensure that N was sufficiently large to allow all possible types of 4-tuples.

To calculate expected value, we counted only the number of 4-tuples containing the edge from 0 to 1, because if N is prime, then drawing any other edge is equivalent to drawing the edge from 0 to 1, by isomorphism. Our expected value calculations make the assumption that N is prime.

The number of completely independent 4-tuples depends on the third vertex chosen. After choosing 0 and 1, there are now three vertices which, if chosen, will force a dependent

edge in a 4-tuple containing 0, 1, and that vertex. These three vertices are 2, $N - 1$, and $\frac{N+1}{2}$. There are then six vertices which, if chosen, will eliminate an additional six vertices. These six 'special' vertices are the four adjacent to the three vertices known to cause a dependence (not including 0 and 1, which are already chosen) and an additional two such that the distance between the two is equal to the distance from 0 to the farthest of the two and is equal to the distance from 1 to the other of the two. For example, on $N = 17$, these two vertices are 6 and 12. Thus, the number of ways to pick an independent 4-tuple containing one of these six 'special' vertices is $(N - 1) * 6 * (N - 12)$. The $(N - 1)$ term comes from the assumption that any vertex chosen is equivalent to choosing 1, by isomorphism. The $(N - 12)$ term comes from the three already chosen vertices, plus the three one-dependencies created by the edge from 0 to 1, plus the six additional vertices that would force a dependency once the first three were chosen. The second way to pick a completely independent 4-tuple is to choose one of another six vertices with odd properties. These six vertices are arranged near the one-dependent vertices such that, when picked they force four two-dependent vertices. All independent points force three two-dependent vertices because of symmetry (for vertex q, the three two-dependent vertices would be $q+1$, $q - 1$, and $N + 1 - q$), but for all graphs with N greater than or equal to 17, there are six vertices that cause another two-dependency that is in fact a crowding of what is usually two one-dependencies. Once one of those 6 vertices is chosen, then eight vertices become unavailable because they will force a dependent edge. Thus, the number of ways to choose an independent 4-tuple in this way is $(N - 1) * 6 * (N - 14)$. After all possible strange third vertex cases have been disregarded, then there remain $(N - 17)$ independent vertices to choose from. When one of these 'normal' vertices has been chosen, then nine vertices become unavailable because they will force a dependent edge. Thus, the number of ways to choose an independent 4-tuple in this way is $(N - 1) * (N - 17) * (N - 15)$. The total expected value of completely independent 4-tuples is

$$(\frac{1}{2})^6((n - 1) * (6) * (n - 12) + (n - 1) * 6 * (n - 14) + (n - 1) * (n - 17) * (n - 15))$$

Similarly, there are four different ways to construct a 4-tuple with one dependent edge, which depend on the third vertex chosen. If the third vertex chosen is the vertex that forces the one dependent edge, then it must be one of the three vertices 2, $N - 1$, or $\frac{N+1}{2}$ that will force a dependent edge with 0 and 1. After one of these three is chosen, then three additional vertices become unavailable, leaving $(N - 9)$ independent vertices. Thus, the number of ways to choose a one-dependent 4-tuple in this manner is $(N - 1) * 3 * (N - 9)$. When the fourth vertex chosen is the one which forces a dependent edge, there are three different ways to choose the third vertex. If the third vertex chosen is one of the first type of strange vertices, the six vertices of which four are adjacent to the three one-dependent

vertices, then five of the remaining vertices force a one-dependent edge. Thus, the number of ways to choose a one-dependent 4-tuple in this manner is $(N-1)*6*5$. If the third vertex chosen is independent and one of the other six strange vertices, then seven of the remaining vertices will force one dependent edge in the 4-tuple. Thus, the number of ways to choose a one-dependent 4-tuple in this manner is $(N-1)*6*7$. If the third vertex chosen is one of the 'normal' independent vertices, then there are nine one-dependent vertices created. Thus the number of ways to choose a one-dependent 4-tuple in this manner is $(N-1)*(N-17)*9$. The expected value of one-dependent 4-tuples is then

$$(\frac{1}{2})^5((n-1)*3*(n-9)+(n-1)*6*5+(n-1)*6*7+(n-1)*(n-17)*9)$$

There are also four different ways to choose a 4-tuple with 2 dependent edges. One way is to choose the third and fourth vertex such that each forces one dependent edge. There are three such vertices for the third vertex, and once one of those three is chosen, then there are four such vertices that could be chosen as the fourth vertex. Thus, there are $(N-1)*3*4$ ways to choose a 2-dependent 4-tuple in this manner. As in previous examples' calculations, there are three different ways to choose an independent third vertex. If the third vertex is one of the six of the first type of strange vertices, then three vertices exist which will force two dependent edges in the 4-tuple. Thus, the number of ways to choose a two-dependent 4-tuple in this manner is $(N-1)*6*3$. If the third vertex chosen is of the other strange type of vertex (one of six, again), then there exist four vertices that would force two dependent edges in the 4-tuple. Thus, the number of ways to choose a two-dependent 4-tuple in this way is $(N-1)*(N-9)*4$. If the third vertex chosen is one of the 'normal' independent vertices, then there are three two-dependent vertices created. Thus the number of ways to choose a one-dependent 4-tuple in this manner is $(N-1)*(N-17)*3$. The expected value of two-dependent 4-tuples is then

$$(\frac{1}{2})^4((n-1)*3*4+(n-1)*6*3+(n-1)*6*4+(n-1)*(n-17)*3)$$

There are two different ways to choose a 4-tuple with three dependent edges, depending, as before, on the third vertex chosen. If the third vertex chosen is one of the three one-dependent vertices created by an edge from 0 to 1, then there exist two vertices which will force two additional edges. Thus, there are $(N-1)*3*2$ ways to choose a three-dependent 4-tuple in this manner. If the third vertex chosen is independent and one of the six vertices either adjacent to 2, $N-1$, or $\frac{N+1}{2}$, or equidistant from each other and the edge from 0 to 1, then there exists one vertex which will force three-dependent edges in the 4-tuple. If the third vertex chosen is independent but not of this type, then no such fourth vertex

exists. Thus, the number of ways to choose a three-dependent 4-tuple with the third vertex independent is $(N - 1) * 6$. The expected value of three-dependent 4-tuples is then

$$(\frac{1}{2})^3 ((N - 1) * 3 * 2 + (N - 1) * 6)$$

To get a true expected value, we must count only distinct 4-tuples. As counted now, we include each set of three vertices, not counting zero, 3! times. And since we can bring any of the other three points to zero by cyclic automorphism, we must also divide out by 4. Thus, our entire equation must be divided by 4! or 24. The complete expected value formula is then

$$(\frac{1}{24})((\frac{1}{2})^6((n - 1) * (6) * (n - 12) + (n - 1) * 6 * (n - 14) + (n - 1) * (n - 17) * (n - 15))$$

$$+(\frac{1}{2})^5((n - 1) * 3 * (n - 9) + (n - 1) * 6 * 5 + (n - 1) * 6 * 7 + (n - 1) * (n - 17) * 9)$$

$$+(\frac{1}{2})^4((n - 1) * 3 * 4 + (n - 1) * 6 * 3 + (n - 1) * 6 * 4 + (n - 1) * (n - 17) * 3)$$

$$+(\frac{1}{2})^3((N - 1) * 3 * 2 + (N - 1) * 6))$$

which simplifies to

$$(\frac{1}{1536})(N - 1)(N^2 + 16N - 9)$$

This expectation matches brute force calculations for all primes from 17 to 43.