ON THE DOMINATION OF KINGS

JOLIE BAUMANN, NEIL CALKIN, AND JEREMY LYLE

Abstract. In this paper, we consider counting the number of ways to place kings on an $k \times n$ chessboard, such that every square is dominated by a king. Let $f(k, n)$ be the number of dominating configurations. We consider the asymptotic behavior of the function $f(k, n)$.

1. Introduction

We consider a $k \times n$ chessboard embedded on a torus, that is, with the first row of the chessboard adjacent to the last row, and the leftmost column adjacent to the rightmost column. We consider the number of different ways to place kings on this chessboard such that every square on the board is dominated; that is, every space on the board either contains a king or can be attacked by one. A particular assignment of kings to the squares on the chessboard is a configuration, and a configuration in which every square is dominated is called a dominating configuration (Figure 1 gives an example of a dominating configuration on a 9×9 chessboard).

Let $f(k, n)$ denote the number of dominating configurations on an $k \times n$ chessboard. In Section 2, we consider counting configurations on chessboards embedded on the torus. We show that the asymptotic behavior of $f(n, k)$ and of the number of dominating configurations on chessboards embedded on the torus is identical. Further, we formulate the counting problem as a probability problem, writing the probability of a configuration being dominating as the intersection of each square being dominated. Dividing the torus into cylindrical bands, we consider the probability of those events contained completely in a cylindrical band, and those events which overlap two bands. A transfer matrix is defined in Section 3 in order to calculate the probability that each event contained in a cylindrical band occurs, and the recursive structure of this matrix is identified. In Section 4, we consider the events which overlap two bands, and obtain bounds on the conditional probability of an event, given the intersection of any collection of events.

This research was partially supported by the NSF grant DMS 0552799.

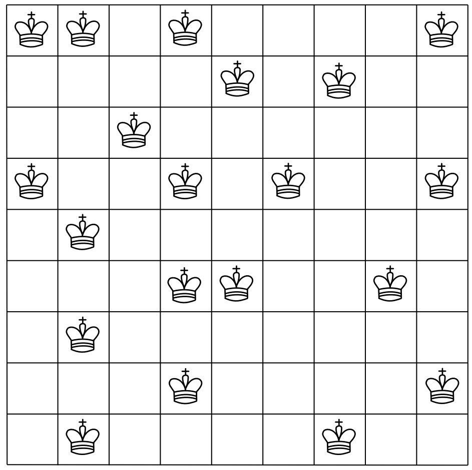

FIGURE 1. A dominating configuration of kings

In Section 5, we combine the two previous sections to determine bounds on the asymptotic behavior of $f(k,n)$. In particular, as in [CW98], we will show that the following double limit exists, (we define its value to be η.)

$$\eta \stackrel{\text{def}}{=} \lim_{n,k} f(n,k)^{\frac{1}{nk}}$$

where $\lim_{n,k} \frac{f(k,n)^{\frac{1}{nk}}}{2} = \lim_{n,k\to\infty} \frac{f(k,n)^{\frac{1}{nk}}}{2}$. In Theorem 5, we give bounds on η, determining that $1.996901390214526 \leq \eta \leq 1.997195304892026$.

In Section 6, we derive an alternative method to attempt to find better bounds on η. To conclude, we briefly discuss the benefits and shortcomings of each method in Section 7.

2. Main Method

2.1. **Formulation.** In order to simplify the problem, we consider counting dominating configurations of kings on a chessboard embedded on a torus. Let $g(k,n)$ be the number of dominating configurations on a $k \times n$ chessboard embedded on a torus, that is, with the first row of the chessboard adjacent to the last row, and the leftmost column adjacent to the rightmost column. Let $\mathcal{S}_t$ be the set of squares on the chessboard on the torus. We uniformly at random choose configurations of kings from the set of all 2^{nk} configurations.

Let A_i be the event that square i is undominated, that is, the 3×3 block of squares, or cell, centered at i does not contain a king. If a square j is located within the 5×5 cell centered at i, the 3×3 cells centered at i and j

overlap, implying dependencies among the events A_i and A_j; however, relative to the number of events, the number of such dependencies is small. It is easy to note that if an event A_i occurs, the configuration is not dominating, so we consider $Pr(\cap_{i\in\mathcal{S}_t}\overline{A_i})$ (in fact, $g(n,k) = 2^{nk}Pr(\cap_{i\in\mathcal{S}_t}\overline{A_i})$).

On the set of all configurations Ω, we define a simple partial ordering as follows: $\omega_1 \prec \omega_2$ for $\omega_1, \omega_2 \in \Omega$ if and only if ω_2 is obtained by adding kings to ω_1. Then for any $S_1 \subseteq \mathcal{S}_t$, if the event $B = \cap_{i\in S_1}\overline{A_i}$ occurs for $\omega_1 \in \Omega$, it also occurs for $\omega_2 \in \Omega$ where $\omega_1 \prec \omega_2$. Events with this property are called superhereditary, and by a result of Kleitman [Kle66], there is positive correlation between any two such events. That is, for sets $S_1, S_2 \subseteq \mathcal{S}_t$, if $B_1 = \cap_{i\in S_1}\overline{A_i}$ and $B_2 = \cap_{i\in S_2}\overline{A_i}$,

$$\Pr(B_1 \cap B_2) \geq \Pr(B_1)P(B_2) \Rightarrow \Pr(B_1|B_2) \geq \Pr(B_1).$$

In particular, $\Pr(\overline{A_i}|\overline{A_j}) \geq \Pr(\overline{A_i})$.

2.2. **Equivalence of Torus and Plane.** Before continuing on to determine bounds on $g(n,k)$, we justify our consideration of the torus, by showing that asymptotically $f(n,k)$ and $g(n,k)$ behave similarly.

Theorem 1. *If* $\lim\limits_{n,k} g(n,k)^{\frac{1}{nk}}$ *exits, then* $\lim\limits_{n,k} f(n,k)^{\frac{1}{nk}}$ *exists, and*

$$\lim_{n,k} g(n,k)^{\frac{1}{nk}} = \lim_{n,k} f(n,k)^{\frac{1}{nk}}$$

Proof. In a similar fashion as the torus, let $\mathcal{S}_p$ be the set of squares on the chessboard embedded in the plane. Let $\partial\mathcal{S}_p \subseteq \mathcal{S}_p$ be the set of squares on the border of the chessboard, and $\partial\mathcal{S}_t \subseteq \mathcal{S}_t$ be a set of squares that would correspond to $\partial\mathcal{S}_p$ if the chessboard was cut along a row and some column, and embedded in the plane. Then let $\mathcal{S}_p^0 = \mathcal{S}_p \backslash \partial\mathcal{S}_p$ and $\mathcal{S}_t^0 = \mathcal{S}_t \backslash \partial\mathcal{S}_t$.

We assume that the $\lim\limits_{n,k} g(n,k)^{\frac{1}{nk}}$ exists. Then,

$$\begin{aligned}\lim_{n,k} g(n,k)^{\frac{1}{nk}} &= \lim_{n,k} \left(2^{nk}\Pr\left(\cap_{i\in\mathcal{S}_t}\overline{A}_i\right)\right)^{\frac{1}{nk}} \\ &= \lim_{n,k} 2\left(\Pr\left(\cap_{i\in\mathcal{S}_t^0}\overline{A}_i\right)\right)^{\frac{1}{nk}}\left(\Pr\left(\cap_{i\in\partial\mathcal{S}_t}\overline{A}_i | \cap_{j\in\mathcal{S}_t^0}\overline{A}_j\right)\right)^{\frac{1}{nk}}\end{aligned}$$

Since the events in $\partial\mathcal{S}_t$ are positively correlated with $\Pr(\overline{A}_i) = (1-2^{-9})$ and the number of events in $\partial\mathcal{S}_t$ is $2n+2k-4$,

$$1 \geq \Pr\left(\cap_{i\in\partial\mathcal{S}_t}\overline{A}_i | \cap_{j\in\mathcal{S}_t^0}\overline{A}_j\right)^{\frac{1}{nk}} \geq \left((1-2^{-9})^{2n+2k-4}\right)^{1/nk}$$

Since the lower bound goes to 1,

$$\lim_{n,k}\Pr\left(\cap_{i\in\partial\mathcal{S}_t}\overline{A}_i | \cap_{j\in\mathcal{S}_t^0}\overline{A}_j\right)^{\frac{1}{nk}} = 1$$

Then $\lim_{n,k} 2 \left(Pr\left(\cap_{i\in\mathcal{S}_t\setminus\partial\mathcal{S}_t}\overline{A}_i\right)\right)^{\frac{1}{nk}}$ exists, and we can have that

$$\lim_{n,k} g(n,k)^{\frac{1}{nk}} = \lim_{n,k} 2\left(\Pr\left(\cap_{i\in\mathcal{S}_t^0}\overline{A}_i\right)\right)^{\frac{1}{nk}}$$

Similarly for the plane, the events in $\partial\mathcal{S}_p$ are positively correlated with $P(\overline{A}_i) > (1-2^{-4})$ (the corners give this value) and the number of events in $\partial\mathcal{S}_p$ is $2n+2k-4$,

$$1 \geq \Pr\left(\cap_{i\in\partial\mathcal{S}_p}\overline{A}_i | \cap_{j\in\mathcal{S}_p^0}\overline{A}_j\right)^{\frac{1}{nk}} \geq \left((1-2^{-4})^{2n+2k-4}\right)^{1/nk}$$

so that

$$\lim_{n,k} \Pr\left(\cap_{i\in\partial\mathcal{S}_p}\overline{A}_i | \cap_{j\in\mathcal{S}_p^0}\overline{A}_j\right)^{\frac{1}{nk}} = 1$$

Also, if events in the "boundaries" of $\mathcal{S}_t$ and $\mathcal{S}_p$ are removed, there is no difference between the plane and the torus, that is

$$\Pr\left(\cap_{i\in\mathcal{S}_t^0}\overline{A}_i\right) = \Pr\left(\cap_{i\in\mathcal{S}_p^0}\overline{A}_i\right).$$

Therefore,

$$\begin{aligned}
\lim_{n,k} f(n,k)^{\frac{1}{nk}} &= \lim_{n,k}\left(2^{nk}\Pr\left(\cap_{i\in\mathcal{S}_p}\overline{A}_i\right)\right)^{\frac{1}{nk}} \\
&= \lim_{n,k} 2\left(\Pr\left(\cap_{i\in\mathcal{S}_p^0}\overline{A}_i\right)\right)^{\frac{1}{nk}}\left(\Pr\left(\cap_{i\in\partial\mathcal{S}_p}\overline{A}_i | \cap_{j\in\mathcal{S}_p}\overline{A}_j\right)\right)^{\frac{1}{nk}} \\
&= \lim_{n,k} 2\left(\Pr\left(\cap_{i\in\mathcal{S}_p^0}\overline{A}_i\right)\right)^{\frac{1}{nk}}\lim_{n,k}\left(\Pr\left(\cap_{i\in\partial\mathcal{S}_p}\overline{A}_i | \cap_{j\in\mathcal{S}_p}\overline{A}_j\right)\right)^{\frac{1}{nk}} \\
&= \lim_{n,k} 2\left(\Pr\left(\cap_{i\in\mathcal{S}_t^0}\overline{A}_i\right)\right)^{\frac{1}{nk}} = \lim_{n,k} g(n,k)^{\frac{1}{nk}} \quad \square
\end{aligned}$$

2.3. **General Approach.** We can partition the squares of the chessboard on the torus into disjoint k/ℓ cylindrical strips, each of width ℓ. For a particular strip, let $B_{\ell,n}$ be the union of each event $\overline{A}_i$ whose corresponding 3×3 cell is completely contained within the strip. This partitions the set of events into two sets, U_1 and U_2, where U_1 is the set of events corresponding to 3×3 cells contained in k/ℓ disjoint cylindrical shells of width ℓ, and U_2 is the set of remaining events.

$$\begin{aligned}
\Pr\left(\cap_{i=1}^{nk}\overline{A}_{\alpha_i}\right) &= \Pr\left(\cap_{i\in U_1}\overline{A}_i\right)\Pr\left(\cap_{i\in U_2}\overline{A}_i | \cap_{i\in U_1}\overline{A}_i\right) \\
&= \left(\Pr(B_{\ell,n})^{k/\ell}\right)\Pr\left(\cap_{i\in U_2}\overline{A}_i | \cap_{i\in U_1}\overline{A}_i\right)
\end{aligned}$$

At a particular boundary between two cylindrical strips, let V_1 be the events corresponding to the two rows on either side of the boundary. We know that for any set $S\subseteq\mathcal{S}_t$,

$$\Pr\left(\cap_{i\in V_1}\overline{A}_i | \cap_{i\in S}\overline{A}_i\right) > \Pr\left(\cap_{i\in V_1}\overline{A}_i\right) = \Pr\left(B_{4,n}\right)$$

We therefore obtain the following upper and lower bounds.

$$\Pr\left(\cap_{i=1}^{nk}\overline{A}_{\alpha_i}\right) \geq \left(\Pr(B_{\ell,n})^{k/\ell}\right)\left(\Pr\left(B_{4,n}\right)^{k/\ell}\right) \tag{1}$$

$$\Pr\left(\cap_{i=1}^{nk}\overline{A}_{\alpha_i}\right) \leq \left(\Pr(B_{\ell,n})^{k/\ell}\right)\Pr\left(\cap_{i\in U_2}\overline{A}_i \middle| \cap_{i\in U_1}\overline{A}_i\right) \tag{2}$$

In Section 3, we use the transfer matrix method to obtain $\Pr(B_{\ell,n})$ for small values of ℓ. In Section 4, we derive upper bounds on the conditional probabilities obtained by expanding the expression $\Pr\left(\cap_{i\in U_2}\overline{A}_i \middle| \cap_{i\in U_1}\overline{A}_i\right)$.

3. The Transfer Matrix

The configurations on the $\ell \times n$ cylindrical strip can be constructed by successively overlaying configurations on $\ell \times 2$ cells of squares. To overlay two configurations, the rightmost column of the previous $\ell \times 2$ cell and the leftmost column of the next $\ell \times 2$ cell must contain the same configuration of kings. To additionally ensure that no 3×3 cell completely contained in the cylindrical strip is empty, we require that any two configurations on a $\ell \times 2$ cell which overlap not contain an empty 3×2 cell at the same location. It is not difficult to see that if $\ell \times 2$ cells are overlayed in this fashion, the result is a configuration in which all squares not located on the top or bottom row are dominated. To count the the number of configurations that can be built up in this way, we use the transfer matrix method (see [Sta97]), a method used on many types of problems, including the related problem of counting independent sets of kings [CW98, CJP$^+$06b, CJP$^+$06a].

To construct the transfer matrix, we first must define the bijection from the configurations on $\ell \times 2$ cells to the matrix indices. Number each of the squares of the $\ell \times 2$ cell by row from the bottom. Then any of the $2^{2\ell}$ configurations on a $\ell \times 2$ cell on a chessboard corresponds to a binary string of length $2k$, where a 1 in position i indicates a king is located in square i, while a 0 indicates there is no king located there (as demonstrated in Figure 2).

The transfer matrix for this problem, T_ℓ ($\ell \geq 3$), is defined in the following way. The matrix T_ℓ has dimension $2^{2\ell} \times 2^{2\ell}$, where both the row i and the column i correspond to the configuration of kings on a $\ell \times 2$ cell with the corresponding binary expansion of i. The entry (i, j) of the matrix T_ℓ is 1 if the configuration on the leftmost column of i coincides with the configuration on the rightmost column of j, and i and j do not contain an empty 3×2 cell at the same location.

The matrix T_ℓ can be viewed as an adjacency matrix of a digraph, where the vertices are the configurations on $\ell \times 2$ cells of squares. A walk of length n beginning at i and ending at i corresponds to a configuration on the $\ell \times n$ cylindrical strip, and the number of such walks is given by $[T_\ell^n]_{i,i}$, the (i, i) entry of T_ℓ^n. Therefore, the total number of configurations on a cylindrical

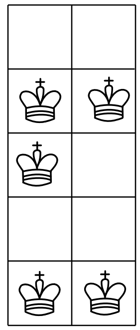

FIGURE 2. The configuration on an 5×2 board corresponding to the binary number 0011010011.

strip of length n and width ℓ such that each event $\overline{A}_i$ occurs is given by $Tr\,(T_\ell^n)$, the trace of T_ℓ^n, i.e.,

$$\Pr\,(B_{\ell,n}) = \frac{Tr\,(T_\ell^n)}{2^{\ell n}}.$$

A nonnegative matrix M is primitive if every entry of M^n is strictly positive for a suitably large value of n. Given any configuration i on a $k \times 2$ cell and any configuration j on a $k \times 2$ cell, a valid sequence of overlaying configurations which start with i and end with j can be easily be found by considering the configuration for which the first two columns match i, the last two columns match j, and all remaining squares contain a king. Therefore, T_ℓ^n is primitive since $T_\ell^4 > 0$. The Perron-Frobenius Theorem describes the dominant eigenvalue in primitive matrices, along with the associated eigenvector.

Theorem 2 (Perron-Frobenius)**.** *Let M be a non-negative, primitive matrix. Then there is a unique eigenvalue λ_r of largest absolute value. Furthermore, $\lambda_r > 0$, and the eigenvector associated with λ_r is positive.*

Let λ_ℓ be the unique principle eigenvalue as guaranteed by the Perron-Frobenius theorem. Since the trace of a matrix is the sum of its eigenvalues, λ_ℓ can be used to describe the asymptotic behavior of $\Pr\,(B_{\ell,n})$ for fixed ℓ and increasing n.

$$\lim_{n\to\infty} \left(\Pr\,(B_{\ell,n})\right)^{1/n} = \lim_{n\to\infty} \left(\frac{Tr(T_\ell^n)}{2^{\ell n}}\right)^{1/n} = \frac{\lambda_\ell}{2^\ell} \tag{3}$$

3.1. **The Recursive Structure of T_ℓ.** The matrix T_ℓ has the nice property of having a simple recursive structure, in terms of $T_{\ell-1}$, $T_{\ell-2}$, and $T_{\ell-3}$, as given below.

$$T_3 = \left[\begin{array}{c}
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\end{array}\right]$$

FIGURE 3. The transfer matrix T_3

$$T_\ell = \begin{bmatrix} S_{\ell-1} & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \\ T_{\ell-1} & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \end{bmatrix}$$

$$S_\ell = \begin{bmatrix} R_{\ell-1} & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \\ T_{\ell-1} & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \end{bmatrix}, \; R_\ell = \begin{bmatrix} 0 & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \\ T_{\ell-1} & T_{\ell-1} & 0 & 0 \\ 0 & 0 & T_{\ell-1} & T_{\ell-1} \end{bmatrix}$$

Note that each 0 represents an appropriately sized block of zeros, and also that the matrices $S_{\ell-1}$, $T_{\ell-1}$, and $R_{\ell-1}$ all have the same dimensions. The matrices T_1 and T_2 are the trivial matrix obtained from only checking to see if two configurations overlap, and the matrix T_3 is given in Figure 3 (Note that the only zero entry that is not a result of non-overlapping configurations is the $(0,0)$ entry).

Theorem 3. *The matrix T_ℓ has the recursive structure given above.*

Proof. Let ω_ℓ denote a configuration on a $2 \times \ell$ cell, and $\omega_{\ell-t}$ denote the configuration on a $2 \times (\ell - t)$ cell formed from removing the last t rows from ω_ℓ. We loosely let $[T_\ell]_{\omega_\ell,\omega'_\ell}$ be the matrix element at the indices corresponding to ω_ℓ and ω'_ℓ.

The matrix T_ℓ can be partitioned into 16 blocks, where the indices of the rows and columns can be partitioned by the four possible configurations on the last row of each configuration on a $2 \times \ell$ cell (which are 00, 01, 10, and 11, 1 corresponding to the placement of a king, and zero corresponding to an empty square.) The zero blocks in the definition of T_ℓ correspond to entries where the last row would not overlap. Therefore we need only to consider configurations ω_ℓ and ω'_ℓ where the last row of ω_ℓ and ω'_ℓ overlap, and we do so by cases.

1. *The last row of either ω_ℓ or ω'_ℓ is nonempty.*
   If the last row of either ω_ℓ or ω'_ℓ is nonempty, then any square in the penultimate row is dominated. Then $[T_\ell]_{\omega_\ell,\omega'_\ell} = 1$ precisely when $[T_{\ell-1}]_{\omega_{\ell-1},\omega'_{\ell-1}} = 1$.
2. *The last row of both ω_ℓ and ω'_ℓ is empty.*
   This case corresponds to the entry $S_{\ell-1}$. Again, we note that the zero blocks in the definition of $S_{\ell-1}$ correspond to entries where the last row (of $\omega_{\ell-1}$ and $\omega'_{\ell-1}$) would not overlap. Again, consider two cases.
   a. *The last row of either $\omega_{\ell-1}$ or $\omega'_{\ell-1}$ is nonempty.*
      In this case the penultimate row of ω_ℓ and ω'_ℓ is dominated, so $[T_\ell]_{i,j} = 1$ precisely when $[T_{\ell-2}]_{\omega_{\ell-2},\omega'_{\ell-2}} = 1$.
   b. *The last row of both $\omega_{\ell-1}$ and $\omega'_{\ell-1}$ is empty.*
      This case corresponds to the entry $R_{\ell-1}$. Again, the zero blocks in the definition of $R_{\ell-1}$ correspond to entries where the last row (of $\omega_{\ell-2}$ and $\omega'_{\ell-2}$) would not overlap. Again, consider two cases.
      i. *The last row of either $\omega_{\ell-2}$ or $\omega'_{\ell-2}$ is nonempty.*
         In this case, the ante penultimate row of ω_ℓ or ω'_ℓ is nonempty, so the penultimate row is dominated. Then $[T_\ell]_{\omega_\ell,\omega'_\ell} = 1$ precisely when $[T_{\ell-3}]_{\omega_{\ell-3},\omega'_{\ell-3}} = 1$.
      ii. *The last row of both $\omega_{\ell-2}$ and $\omega'_{\ell-2}$ is empty.*
         In this case the last three rows of both ω_ℓ and ω'_ℓ are empty, so the common square in the penultimate row is not dominated, so $[T_\ell]_{\omega_\ell,\omega'_\ell} = 0$

Thus, the matrix T_ℓ can be defined recursively by the formula given. □

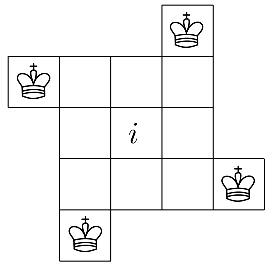

FIGURE 4. The configuration of kings on the removed section.

4. An Upper Bound on the Conditional Probability

In Equation 2, we would like to bound the term $\Pr\left(\cap_{i\in U_2}\overline{A}_i \middle| \cap_{i\in U_1} \overline{A}_i\right)$ from above. We do this by finding bounds on the expression

$$\max_{S\subseteq \mathcal{S}_t-\{i\}} \Pr\left(\overline{A}_i \middle| \cap_{i\in S} \overline{A}_i\right)$$

Trivially, this is bounded above by 1. Slightly more work achieves a bound of $(1-2^{-13})$. Consider the configuration of squares given in Figure 4. For $S \in \mathcal{S}_t$ ($i \notin S$) and $B = \cap_{j\in S}\overline{A}_j$, any configuration where both B and $\overline{A}_i$ occur can be modified by removing thirteen squares in this pattern, and replacing them with the configuration given in Figure 4, so that B still occurs, and $\overline{A}_i$ does not. Then among every 2^{13} configurations for which the event B occurs, the event A_i occurs in at least one, so $\Pr(\overline{A}_i|B) < 1-2^{-13}$.

However, we can improve on this by proving a positive correlation version of the Local Lemma. This approach follows the proof of the Local Lemma given in Chapter 5 of [AS00].

Let T_i be a subset of the events which are dependent on the event A_i. We choose x_i for each event A_i such that

$$\Pr(A_i \cap_{j\in T_i} \overline{A}_j) \geq x_i \prod_{j\in T_i} (1-x_j) \tag{4}$$

for all $1 \leq i \leq nk$.

Lemma 1. *For any set* $S \subseteq \mathcal{S}_t$,

$$\Pr\left(A_i \middle| \cap_{j\in S} \overline{A}_j\right) \geq x_i$$

Proof. We use induction on the cardinality of the set S. We partition S into sets S_1 and S_2, such that S_1 is the set of events in S which are dependent on the event A_i, and the set S_2 is composed of the remaining events of S.

$$\Pr\left(A_i \middle| \cap_{j\in S} \overline{A}_j\right) = \frac{\Pr\left(A_i \cap \left(\cap_{j\in S_1}\overline{A}_j\right) \middle| \left(\cap_{j\in S_2}\overline{A}_j\right)\right)}{\Pr\left(\cap_{j\in S_1}\overline{A}_j \middle| \cap_{j\in S_2} \overline{A}_j\right)} \tag{5}$$

Consider the chessboard with the 3×3 cell centered at i removed. The events $\left(\cap_{j\in S_1}\overline{A}_j\right)$ and $\left(\cap_{j\in S_2}\overline{A}_j\right)$ are still positively correlated on the remaining board, yielding the following equation.

$$\Pr\left(\left(\cap_{j\in S_1}\overline{A}_j\right)\cap\left(\cap_{j\in S_2}\overline{A}_j\right)|A_i\right)\geq \Pr\left(\left(\cap_{j\in S_1}\overline{A}_j\right)|A_i\right)\Pr\left(\left(\cap_{j\in S_2}\overline{A}_j\right)|A_i\right)$$

Thus,

$$\begin{aligned}\Pr(A_i\cap(\cap_{j\in S_1}\overline{A}_j)|(\cap_{j\in S_2}\overline{A}_j)) &> \frac{\Pr(A_i\cap(\cap_{j\in S_1}\overline{A}_j))\Pr(A_i\cap(\cap_{j\in S_2}\overline{A}_j))}{\Pr(A_i)\Pr\left(\cap_{j\in S_2}\overline{A}_j\right)}\\ &= \Pr\left(A_i\cap\left(\cap_{j\in S_1}\overline{A}_j\right)\right)\\ &\geq x_i\prod_{j\in S_1}(1-x_j)\end{aligned}\tag{6}$$

Expand the denominator, and apply the induction hypothesis to find

$$\begin{aligned}\Pr\left(\cap_{j\in S_1}\overline{A}_j|\cap_{j\in S_2}\overline{A}_j\right) &= \Pr\left(\overline{A}_{j_1}|\cap_{j\in S_2}\overline{A}_j\right)\Pr\left(\overline{A}_{j_2}|A_{j_1}\cap\left(\cap_{j\in S_2}\overline{A}_j\right)\right)\\ &\quad\ldots\Pr\left(\overline{A}_{j_r}|\cap_{i=1}^{r-1}A_{j_i}\cap\left(\cap_{j\in S_2}\overline{A}_j\right)\right)\end{aligned}$$

$$\begin{aligned}&=\left(1-\Pr\left(A_{j_1}|\cap_{j\in S_2}\overline{A}_j\right)\right)\left(1-\Pr\left(A_{j_2}|A_{j_1}\cap\left(\cap_{j\in S_2}\overline{A}_j\right)\right)\right)\\ &\quad\ldots\left(1-\Pr\left(A_{j_r}|\cap_{i=1}^{r-1}A_{j_i}\cap\left(\cap_{j\in S_2}\overline{A}_j\right)\right)\right)\\ &\leq(1-x_{j_1})(1-x_{j_2})\ldots(1-x_{j_r})\\ &=\prod_{j\in S_1}(1-x_j)\end{aligned}\tag{7}$$

Applying the upper bounds on the denominator from (6) and the lower bounds on the numerator from (7) to (5) yields the desired result. □

In order to determine x_i, we define $\overline{B}_j$ to be the event that there is a king in the 3×3 block of squares centered at j, minus any squares contained in the 3×3 block of squares centered at i. Again, these events are positively correlated and independent of A_i.

$$\begin{aligned}&\Pr\left(A_i\cap_{j\in T_i}\overline{A}_j\right)\\ &\quad\geq \Pr(A_i)\Pr\left(\cap_{j\in T_i}\overline{B}_j\right)\\ &\quad\geq \Pr(A_i)\prod_{j\in T_i}\Pr(\overline{B}_j)\\ &\quad=(1-2^{-9})(1-2^{-3})^{a_1}(1-2^{-5})^{a_2}(1-2^{-6})^{a_3}(1-2^{-7})^{a_4}(1-2^{-8})^{a_5}\end{aligned}$$

where $a_1,a_2,a_3,a_5\leq 4$, $a_4\leq 6$, the values dependent on the events in the set T_i. In fact, the events corresponding to the a_1 and a_3 exponents overlap, so the condition $a_1 \leq 4 - a_3$ can be added. We can then choose $x_i = 0.001150168444385$ for all i, which gives

$$\max_{S \subseteq \mathcal{S}_t - \{i\}} \Pr\left(\overline{A}_i | \cap_{i \in S} \overline{A}_i\right) \leq 0.99884983155562. \tag{8}$$

5. Results

The function $g(n,k)$ is submultiplicative, so $\lim_{n\to\infty} g(n,k)^{1/nk}$ exists for every k. Taking the limit as $n \to \infty$ of the nk-th root of both sides of Equations 2 and 1, and applying Equation 3, we obtain

$$\lim_{n\to\infty} \left(\Pr\left(\cap_{i=1}^{nk} \overline{A}_{\alpha_i}\right)\right)^{1/nk} \geq \left(\frac{\lambda_\ell}{2^\ell}\right)^{1/\ell} \left(\frac{\lambda_4}{2^4}\right)^{1/\ell} \tag{9}$$

$$\lim_{n\to\infty} \left(\Pr\left(\cap_{i=1}^{nk} \overline{A}_{\alpha_i}\right)\right)^{1/nk} \leq \left(\frac{\lambda_\ell}{2^\ell}\right)^{1/\ell} (0.99884983155562)^{2/\ell} \tag{10}$$

Since $\lambda_6 = 63.6096223408566$ and $\lambda_4 = 15.9491257240594$, we obtain

$$0.99845069510726 \leq \lim_{n\to\infty} \left(\Pr\left(\cap_{i=1}^{nk} \overline{A}_{\alpha_i}\right)\right)^{1/nk} \leq 0.99859765244601$$

Equations 9 and 10 also allow us to show that $\lim\limits_{n,k} g(n,k)^{\frac{1}{nk}}$ exists.

Theorem 4. *The limit* $\lim\limits_{n,k} g(n,k)^{\frac{1}{nk}}$ *exists.*

Proof. We show that $\lim\limits_{n,k} \left(\Pr\left(\cap_{i=1}^{nk} \overline{A}_{\alpha_i}\right)\right)^{1/nk}$ exists. Using circular strips of width s for in Equation 10 and width t in Equation 9, we can achieve the following inequality.

$$\lambda_s^{1/s} > \lambda_t^{1/t} \left(\frac{\lambda_4}{16}\right)^{1/t}$$

This inequality is true for any choice of s, so we may take the $\liminf$ as $s \to \infty$ of the left,

$$\liminf \lambda_s^{1/s} > \lambda_t^{1/t} \left(\frac{\lambda_4}{16}\right)^{1/t}$$

Taking the $\limsup$ as $t \to \infty$ of both sides yields

$$\liminf \lambda_s^{1/s} > \limsup \lambda_t^{1/t}.$$

Therefore, the limit $\lim_{\ell\to\infty} \lambda_\ell^{1/\ell}$ exists, which implies that the following double limit exists

$$\eta_p \stackrel{\text{def}}{=} \lim_{n,k} P\left(\cap_{i=1}^{nk} \overline{A}_{\alpha_i}\right)^{1/nk}$$ □

Applying Theorem 1 gives the following theorem.

Theorem 5.

$$1.996901390214526 \leq \eta \leq 1.997195304892026$$

6. An Alternative Approach

Lastly, we consider an alternative approach to counting the number of dominating configurations of kings on an $k \times n$ chessboard. Instead of choosing random configurations uniformly from the space of all configurations, choose random configurations from the space of all configurations with M kings. Let $A = nk$ to simplify the expressions to follow. Let Ah_M denote the expected number of kings needed to "fix" a configuration, that is, the expected value of the minimum number of kings which need to be added to a random configuration with M kings, such that the configuration can be made dominating.

Remove Ah_M kings from each dominating configurations with $Ah_M + M$ kings. This leaves configurations with M kings. At least $1/Ah_M$ of the configurations with M kings require the addition of Ah_M or less kings in order to become a dominating configuration. Thus removing Ah_M kings from the set of all dominating configurations covers at least $1/Ah_M$ of the number of configuration with exactly M kings.

$$f(k,n)\binom{M + Ah_M}{Ah_M} \geq \frac{1}{Ah_M}\binom{A}{M} \tag{11}$$

$$f(k,n) \geq \frac{1}{Ah_M}\frac{\binom{A}{M}}{\binom{M+Ah_M}{Ah_M}}$$

We note that this is valid for any function greater than h_M, as long as $M + Ah_M < A$. To simplify things, we let $M = pA$ for an appropriate value of p, and also set $q = 1 - p$. Expanding the righthand side and making these substitutions, we obtain the following.

$$f(k,n) > \frac{1}{Ah_M}\frac{\binom{A}{pA}}{\binom{pA+Ah_M}{Ah_M}} = \frac{1}{Ah_M}\frac{A!(Ah_M)!}{(qA)!(pA+Ah_M)!}.$$

Apply Stirling's Approximation (we over-estimate the error term with e) to obtain

$$f(k,n) > \frac{1}{Ah_M}\sqrt{\frac{h_M}{q(p+h_M)}}\frac{A^A(Ah_M)^{Ah_M}}{(qA)^{(qA)}(pA+Ah_M)^{(pA+Ah_M)}}e.$$

Taking the A-th root of both sides and simplifying yields

$$f(n,k)^{1/A} \geq \left(\frac{1}{Ah_M}\left(\frac{h_M}{q(p+h_M)}\right)^{1/2} e\right)^{1/A} \frac{{h_M}^{h_M}}{q^q(p+h_M)^{(p+h_M)}}.$$

As $n, k \to \infty$, this equation becomes

$$\eta \geq \frac{{h_M}^{h_M}}{q^q(p+h_M)^{(p+h_M)}}.$$

6.1. **Approximating h_M.** In order to approximate the function h_M, we instead consider placing kings at squares on the board independently with uniform probability p. In order to examine h_M, we choose p such that $M = pA$, as we chose previously. In this distribution, we again consider the expected number of kings needed to "fix" a configuration, and we will let this value be Ah_p. The quantities h_p and h_M are related if we consider $h_{p'}$ and let $p' \to p$, as we will show below.

Let $\Omega = \mathcal{P}([n])$, the power set of $\{1, 2, \ldots, n\}$. Let $E_p(X)$ be the expected value of a random variable X on a subset $S \subset [n]$ such that the elements of S are chosen uniformly at random from $[n]$ with probability p. Let $E^*_k(X)$ be the expected value of a random variable X on a subset S with k elements, such that S is chosen uniformly at random from all k-element subsets. We assume that the quantity $pn = M$ is an integer. As discussed in [Bol01], the quantities E_p and E^*_M are closely related, and we give one aspect of that relationship below.

Lemma 2. *Suppose p, X, and S are chosen as above, with $p > 1/2 + \varepsilon$ for $\varepsilon > 0$, $E^*_k(X), E_p(X) > 0$ for all k, p, and $E^*_k(X) < E^*_{k'}(X)$ for $k' < k$. Then for $p_1 = p - \sqrt{\frac{pq}{n}}$ and large enough values for n,*

$$E_{p_1}(X) > E^*_{M=pA}(X)$$

Proof. By splitting the elements in Ω by subset size, we can write $E_{p_1}(X)$ as follows:

$$E_{p_1}(X) = \sum_{k=1}^{n} \binom{n}{k} E^*_k(X) p_1^k q_1^{n-k}$$

We define $p_2 = p_1 - \sqrt{\frac{pq}{n}}$ (We will treat $p_2 n$ as an integer, avoiding floors and ceilings in the summations which do not change the result), in order to consider a thin slice of this sum centered at p. Since all terms of this sum are positive, we obtain

$$E_{p_1}(X) \geq \sum_{k=p_2 n}^{p_1 n} \binom{n}{k} E^*_k(X) p_1^k q_1^{n-k}.$$

Using the fact that $E^*_M(X) < E^*_k(X)$ for all $p_2 n \leq k \leq p_1 n$, as well as the fact that since p, p_1, p_2 are larger than $1/2$, $p_1^k q_1^{n-k} > p_2^{p_2 n} q_2^{q_2 n}$, we obtain

$$\begin{aligned} E_{p_1}[X] &\geq E^*_M(X) \sum_{k=p_2 n}^{p_1 n} \binom{n}{k} p_1^k q_1^{n-k} \\ &\geq E_M(X)(p_1 n - p_2 n) \binom{n}{p_2 n} p_2^{p_2 n} q_2^{q_2 n} \end{aligned}$$

Using Stirling's approximation, we obtain the following

$$\begin{aligned} E_{p_1}(X) &> E^*_M(X)\left(\frac{p_1 n - p_2 n}{\sqrt{p_2 q_2 n}}\right) \\ &= E^*_M(X)\sqrt{\frac{pq}{p_2 q_2}} > E^*_M(X) \quad \square \end{aligned}$$

Therefore, we will consider h_p as an upper bound on h_M, as we approach this value as $A = nk$ grows larger.

An upper bound on the number of kings needed to "fix" a configuration is the number of undominated squares. If kings are placed uniformly at random with probability p (with $q = 1-p$), the expected number of squares which are not dominated $q^9 A$, so $h_p \leq q^9$.

We can do better at approximating h_p using inclusion/exclusion. A king placed at the center of an empty 3×3 cell will also fix the problem of having the 3×3 cell directly above empty as well. Let X be the random variable which determines the number of empty rectangular cells of area 12 or less with length and width at least 3 squares. We can count this by counting the number of empty 3×3 cells, as well as the number of empty 3×4 and 4×3 cells, and then using inclusion and exclusion to represent when an empty 3×3 cell is contained in the intersection of some collection of empty 3×4 and 4×3 cells. The expected value of X is

$$\begin{aligned} E\left(X\right) &= A\left[(q^9 + 2q^{12}) - 4q^{12} + 4q^{15} - 4q^{18} + q^{21}\right] \\ &= A\left[q^9 - 2q^{12} + 4q^{15} - 4q^{18} + q^{21}\right] \end{aligned}$$

This gives a smaller upper bound on the value of h_p. We can apply the same process to find tighter bounds on h_p, but further inclusion/exclusion does not change the coefficients of the q^9 and q^{12} terms.

We can refine this approach by noticing that multiple placements of kings will "fix" a configuration. In particular, if there is an empty 3×3 cell, a king placed on any of the nine squares dominates the center square. In our particular case, we have approximated h_p by counting the number of empty 3×4 and 4×3 cells, as well as the number of empty 3×3 which are not contained in any empty 3×4 and 4×3 cells. A king can be placed on any of six squares to fix each of these empty squares. In Equation 11, if the quantity Ah_M is taken to be the expected number of empty 3×4, 4×3, and 3×3 cells as described above, then we can say the following.

$$f(k,n) \geq 6^{Ah_M}\frac{1}{Ah_M}\frac{\binom{A}{M}}{\binom{M+Ah_M}{Ah_M}} \tag{12}$$

This yields the following.

$$\eta \geq 6^{h_M}\frac{{h_M}^{h_M}}{q^q(p+h_M)^{(p+h_M)}}$$

This gives the lower bound $\eta > 1.98653$ at $p \approx .520735$; however, the lower bound obtained by the transfer-matrix method is tighter, however we feel that improvements to this method may lead to better lower bounds.

7. Conclusion

Both the main method used and the alternative method discussed in the previous section have advantages and disadvantages. Further refinement of the bounds in Theorem 5 is certainly likely using the main method and eigenvalues of larger transfer matrices, but this is very computationally intensive. Additionally, unless the recursive structure of the transfer matrices can be translated into a recursive structure of their respective maximum eigenvalues, an exact answer seems unlikely to result from this approach.

The alternative approach from the previous section does not give as good a bound on η, but we feel that some method based off this approach is much more likely to result in an exact solution then something based on the transfer matrix method. At the moment, it is unclear how close our estimated value of h_p is to the true solution, and a better approximation to this may dramatically improve the bound resulting from this approach.

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(Baumann) Department of Psychology, Northeastern University, Boston, MA 02115 USA

(Calkin) Department of Mathematical Sciences, Clemson University, Clemson, SC 29634 USA
E-mail address: calkin@clemson.edu

(Lyle) Department of Mathematics, The University of Southern Mississippi, 118 College Drive, Hattiesburg, MS 39406-0001 USA
E-mail address: samuel.lyle@usm.edu