Dominating Configurations of Kings

Jolie Baumann

July 11, 2006

Abstract

In this paper, we are counting natural subsets of graphs subject to local restrictions, such as counting independent sets of vertices, dominating sets of vertices, and independent sets of edges. We will discuss the following counting problems: The number of configurations in each problem grows rapidly, leading to the development of bounds and a discussion of entropy.

1 Introduction

In this section of our paper, we explore the number of different ways to place non-independent kings on a chessboard with infinite dimensions such that every space on the board is dominated; that is, every space on the board either contains a king or can be attacked by one. Any configuration of kings that meets these conditions is called a dominating configuration. In our dominating configurations we allow non-independent or non-attacking kings; that is, we allow configurations where two kings can be placed adjacent to one another. First, we define a recursive formula for generating adjacency matrices for dominating configurations of kings on chessboards with various numbers of rows. Next, we develop rough bounds for the number of different dominating configurations on a kxn board. Finally, we work out upper and lower bounds for an entropy constant that tells us how quickly the number of dominating configurations is growing as the values of k and n increase towards infinity.

The kings problem has been researched extensively. Although we have not found any previous work related to dominating configurations of kings, a variety of articles on counting independent sets of kings proved useful in guiding our thoughts and approaches regarding dominating sets.

2 Developing a Recursive Definition

In order to study the number of dominating configurations on a k by n chessboard, we create adjacency matrices. First, we construct boards of size k by 2, one for each possible configuration of kings on a k by 2 chessboard (since we are allowing non-attacking kings, all possible configurations are acceptable).

Labeling Configurations:

In order to keep our adjacency matrices consistent, we create a labeling system that designates a number to each possible configuration of kings on a k by 2 board. We label each board using binary code in such a way that the first two places in the binary number refer to the top two squares (the top row) of a board, the second two places in the binary number refer to the second two squares (the second row) of a board, and so on. A zero designates a blank square and a 1 designates a king in the corresponding square of a board. For example, the binary number 0011010011 would designate the 5 by 2 board:

K	K
	K
K	K

The binary numbers range from 0 (which designates the blank board) to 2^{2k}-1 (which designates a k by 2 board filled with kings).

Overlapping Boards:

Although we vary the number of rows k, the boards always need to be 2 columns wide because any given spot can be dominated from a column ahead or a column behind. To take this into account, we create our adjacency matrices considering only overlapping boards; that is, for any row i there will only be a 1 in the i^{th} j^{th} entry of the matrix if the left hand column of j is identical to the right hand column of i AND this overlapping column is dominated. Therefore, in our adjacency matrices we are considering boards of size k by 3 and we are only concerned with domination in the middle column. We are not concerned with the outer 2 columns being dominated because they could be dominated from a column ahead or a column behind. These outer columns will be the middle column in other overlapping boards, and will be taken care of at other entries in the matrix.

Let A_k refer to the overlapping adjacency matrix for boards with k rows. Because all possible configurations are considered, A_k is always a square matrix of the size 2^{2k} by 2^{2k}.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\ 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \end{pmatrix}$$

Since, A_k grows very quickly as k increases, it is important to develop a recursive definition for finding A_k for large values of k. To do this, we first consider a simpler set of matrices, denoted B_k.

An Important Side Note:

Before we begin, it is important to note that the following recursive definition will not hold for small values of k. The first adjacency matrices are unusual because the boards being considered are so small. For instance, on a 2 by 2 board all of the spots will be dominated if there is a king anywhere on the board, giving us a zero only in the 00 entry. Therefore, any recursion that we can define for A_k will not make sense until $k > 2$. Although we will define some other matrices recursively starting at values of $k \leq 2$, it is important to remember that the recursion will not hold for A_k until k = 3. Now, on to generating a recursion:

Let B_k refer to a matrix of size 2^{2k} by 2^{2k}. If i and j overlap (the right hand column of i is identical to the left hand column of j) then there will be a 1 in the $i^{th}j^{th}$ entry of B_k. It is easy to work with B_k because it follows a simple recursion:

$$B_1 = \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 1&1&0&0 \\ 0&0&1&1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} B_1 & B_1 & 0 & 0 \\ 0 & 0 & B_1 & B_1 \\ B_1 & B_1 & 0 & 0 \\ 0 & 0 & B_1 & B_1 \end{pmatrix}$$

And in General

$$B_k = \begin{pmatrix} B_{k-1} & B_{k-1} & 0 & 0 \\ 0 & 0 & B_{k-1} & B_{k-1} \\ B_{k-1} & B_{k-1} & 0 & 0 \\ 0 & 0 & B_{k-1} & B_{k-1} \end{pmatrix}$$

Next, we consider another set of matrices of the size 2^{2k} by 2^{2k} which also follow a nice recursion. We will denote this set of matrices by E_k since it generates much of the error term between B_k and A_k.

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} E_2 & E_2 & 0 & 0 \\ 0 & 0 & E_2 & E_2 \\ E_2 & E_2 & 0 & 0 \\ 0 & 0 & E_2 & E_2 \end{pmatrix}$$

And in General

$$E_k = \begin{pmatrix} E_{k-1} & E_{k-1} & 0 & 0 \\ 0 & 0 & E_{k-1} & E_{k-1} \\ E_{k-1} & E_{k-1} & 0 & 0 \\ 0 & 0 & E_{k-1} & E_{k-1} \end{pmatrix}$$

By subtracting E_k from B_k we take care of most of the zeros that appear in A_k in positions that overlap but fail to dominate. However, we must consider another set of matrices that can be defined recursively to account for a growing block of zeros in the top left corner of our overlapping adjacency matrices.

Because of the way we labeled our k by 2 boards, any boards designated by lower numbers will only have kings clumped in the bottom rows, thus failing to dominate the upper rows. Therefore, as k increases we must reach larger and larger numbers before all rows can be dominated, resulting in a growing block of zeros in the upper left hand corner of A_k. Because the top rows of the board must be dominated by a king in one of the top 4 squares, the growing block of zeros is of the size 2^{2k-4} by 2^{2k-4}.

Let J_k be a matrix of ones of the size 2^{2k-4} by 2^{2k-4}. We can think of J_k as being defined recursively:

$$J_2 = \begin{pmatrix} 1 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

And in General

$$J_k = \begin{pmatrix} J_{k-1} & J_{k-1} & J_{k-1} & J_{k-1} \\ J_{k-1} & J_{k-1} & J_{k-1} & J_{k-1} \\ J_{k-1} & J_{k-1} & J_{k-1} & J_{k-1} \\ J_{k-1} & J_{k-1} & J_{k-1} & J_{k-1} \end{pmatrix}$$

So that we can use matrix addition and subtraction we need to define J_k within a larger matrix of all zeros. Let D_k be a matrix of size 2^{2k} by 2^{2k} such that:

$$D_2 = \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} J_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And in General

$$D_k = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix}$$

When we subtract out D_k from B_k-E_k we do not get the desired grid of zeros in the top left corner of A_k; some negative numbers appear because E_k and D_k compensate for the same zeros in this top corner. To fix this, we create a final matrix of size 2^{2k} by 2^{2k} with all zero entries outside the top left corner, denoted C_k:

C_3 = a 64 x 64 zero matrix with a 1 in the C_{00} entry

$$C_4 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

And in General

$$C_k = \begin{pmatrix} E_{k-2} & 0 \\ 0 & 0 \end{pmatrix}$$

Our final formula for A_k, based on the recursive definitions of B_k, E_k, D_k, and C_k is:

$$A_k = B_k - E_k - D_k + C_k$$

3 Calculating Bounds for Dominating Configurations

Let $f(k,n)$ denote the number of dominating configurations on a k by n chessboard. We can calculate $f(k,n)$ by summing the entries in A_k. However, we know a dominated chessboard can not begin or end with a blank k by 2 board, so we must ignore the zero row and the zero column in our sums. For this section on dominating kings, whenever we refer to summing the entries of a matrix we are not including the entries in the zero row or zero column in our sum.

Thinking Graphically:

It may be easier at this point to think of the possible boards as vertices in a graph. There is an edge between vertices i and j (a one in the $i^{th}j^{th}$ entry of the adjacency matrix) if the boards i and j overlap and dominate. Therefore, we can refer to a dominated chessboard made up of n columns as a walk of length n around the graph. In terms of the kings problem, this would be all dominating configurations with n columns that include somewhere the overlapping boards i and j.

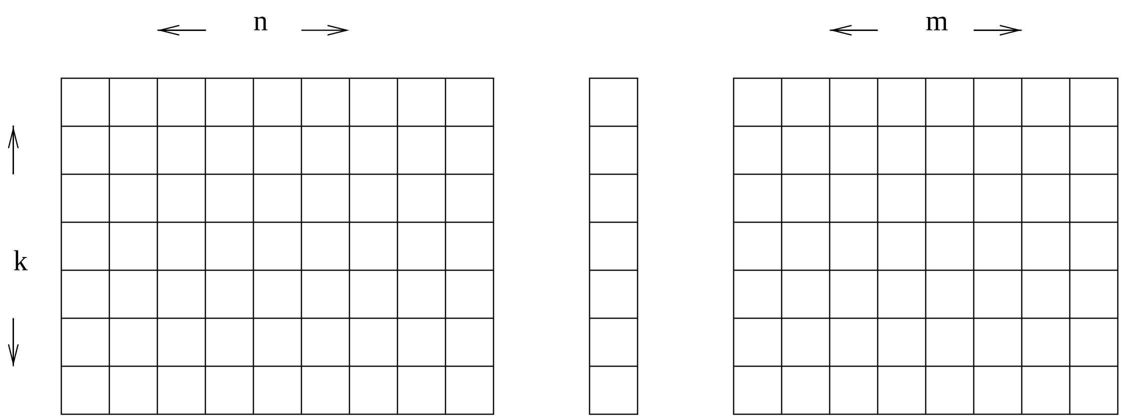
The number in the $i^{th}j^{th}$ entry of the adjacency matrix refers to the number of walks of the graph that use the edge connecting i and j. When we take powers of A_k, we get walks of greater and greater length. Because A_k is made up of two k by 2 boards with an overlapping column, the sum of the entries in A_k gives us the number of dominating configurations for a k by 3 chessboard (the number of walks of length 3). Summing the entries in A_k^2 gives us the number of dominating configurations for a k by 4 board (the number of walks of length 4). In general, the sum of the entries in A_k^n will give us the number of dominating configurations on a k by $n+2$ chessboard (the number of walks of length $n+2$).

Walks of Length n

n	k = 1	k = 2	k = 3	k = 4	k = 5
5	17	891	29,789	963,657	30,947,346
10	355	857,871	871,651,255	$8.9410 * 10^{11}$	$9.1881 * 10^{14}$
15	7,473	826,346,529	$2.5505 * 10^{13}$	$8.2959 * 10^{17}$	$2.7281 * 10^{22}$
20	157,305	$7.9598 * 10^{11}$	$7.4632 * 10^{17}$	$7.6974 * 10^{23}$	$8.1000 * 10^{29}$
100	$7.9067 * 10^{26}$	$4.3727 * 10^{59}$	$2.1557 * 10^{89}$	$2.3229 * 10^{119}$	$2.9551 * 10^{149}$
200	$2.3064 * 10^{53}$	$2.0680 * 10^{119}$	$4.5646 * 10^{178}$	$5.1956 * 10^{238}$	$8.3787 * 10^{298}$

3.1 Adding Dominated Chessboards Together

As you can see, the number of dominating configurations for a k by n board increases rapidly as k and n increase towards infinity. Therefore, it will be helpful to consider other ways of calculating $f(k,n)$ for large values of k and n. For instance, take two dominated chessboards which both have k rows, one with n columns and the other with m columns. If we add a dominated column (with k rows) inbetween these two boards, then we can create a larger chessboard that must also be dominated. Because the column dominates itself, it should be obvious that the newly created board of size k by $(n + m + 1)$ must also be dominated. The following visual might be of use:



Because it is difficult to find the exact number of dominating configurations as k increases, it is useful to find bounds for $f(k, n + m + 1)$.

3.1.1 Upper Bound:

Keeping in mind that $f(k, n) * f(k, m) * f(k, 1) = f(k, n + m + 1)$, we can find bounds for $f(k, n + m + 1)$ by bounding $f(k, 1)$. We know that the total possible number of ways to place kings in a column with k rows is 2^k. Because the total number of configurations must be greater than or equal to the number of configurations that dominate, it follows that:

$$f(k, n + m + 1) \leq f(k, n + m)2^k$$

This gives us a nice upper bound for $f(k, n + m + 1)$.

3.1.2 Lower Bound:

To get a lower bound, we must find δ such that:

$$f(k, n) * f(k, m) * \delta \leq f(k, n + m + 1)$$

To do so, we must first introduce some notation regarding the single column we are adding in. Let:
k = number of rows in the single column
a_k = total number of dominating configurations in the column
b_k = number of dominating configurations with a king in the bottom square
c_k = number of dominating configurations without a king in the bottom square
It should follow quite simply that:

$$a_k = b_k + c_k$$

We also know that the last two squares of any column can not both be blank if we want it to be dominated by itself. In the case of the single column, therefore, any dominating configuration that does not have a king in the last square must have a king in the square directly before it. It follows that:

$$c_n = b_{n-1}$$

Also, we know that if there is a king in the final square, then it does not matter whether or not there is a king in the square directly before it as long as there is a king within the previous 3 spaces. In other words,

$$b_n = a_{n-1} + b_{n-3}$$

Using the above three equations, we can easily prove that a_k is the k^{th} tribonacci number.

$$\begin{aligned}
a_k &= b_k + c_k \\
&= a_{k-1} + b_{k-3} + b_{k-1} \\
&= b_{k-1} + c_{k-1} + b_{k-3} + b_{k-1} \\
&= a_{k-2} + b_{k-4} + c_{k-1} + b_{k-3} + b_{k-1} \\
&= a_{k-2} + c_{k-3} + c_{k-1} + b_{k-3} + b_{k-1} \\
&= (b_{k-1} + c_{k-1}) + a_{k-2} + (b_{k-3} + c_{k-3}) \\
&= a_{k-1} + a_{k-2} + a_{k-3}
\end{aligned}$$

Initial Conditions:

It is simple to get the initial conditions for a_k by hand:
-For a_0, there is only one way to dominate no squares – having no squares. So, $a_0 = 1$.
-For a_1 consider adding a square to nothing. There are only two ways to add a block (add a blank square or add a king). Therefore, there is only one dominating configuration and $a_0 = 1$.
-For a_2 consider adding a square to a single square. This time, lets calculate b_k and c_k separately. The only way to have a blank square at the end of two squares and still have

both squares dominated is to have a king in the first square. Therefore $c_k = 1$. On the other hand, if there is a king in the second square, there are two options for the first square: either it has a king or it doesn't. Because the king in the second square will dominate both squares independent of the first square, we have $b_k = 2$. Since we know $a_k = b_k + c_k$, it follows that $a_2 = 3$.

Thus, the initial conditions are the first terms of the tribonacci sequence:
$a_0 = 1$
$a_1 = 1$
$a_2 = 3$

Therefore, one estimate for δ that gives us a lower bound is

$$\delta = a_k$$

where a_k denotes the k^{th} tribonacci number.

4 Calculating Bounds for an Entropy Constant

Another way of thinking about how rapidly the number of dominating configurations increases as k and n increase towards infinity is to calculate an entropy constant. An entropy constant will help us grasp how fast the number of dominating configurations is growing and if there are bounds on its rate of increase. The entropy constant η gives us information on what percentage of all possible boards have valid dominating configurations of kings. To calculate η we look at $f(k,n)^{\frac{1}{kn}}$ as n and k increase towards infinity. Because there is not a simple way to calculate η we will again search for upper and lower bounds.

To calculate bounds for the entropy constant we must imagine that the chessboard is a torres. That is, a chessboard where the top of each column wraps around and connects to the bottom of each corresponding column, and the right side of each row wraps around and connects to the left side of each corresponding row. When imagining a torres, it is usually helpful to envision a donut.

Next, because any 3x3 block of squares is dominated if there is a king in the center, we can imagine tiling the chessboard with overlapping 3x3 blocks. Interestingly, domination in any given block of 3x3 is only affected by the 24 overlapping blocks of 3x3 immediately around it; the probability of it being dominated is independent of the rest of the chessboard.

4.1 Lovasz Local Lemma

According to Lovasz Local Lemma:

$$Pr(A_i) \leq x_i \prod_{j}^{m} (1 - x_j)$$

and

$$Pr(\wedge \bar{A}_m) \geq \prod_{i=1}^{m} (1 - x_i)$$

Notation:

$Pr(A_i)$ refers to the probability that a given event A_i will occur. In our case, $Pr(A_i)$ is the probability that no kings will appear in a given 3x3 block. So, $Pr(A_i) = \frac{1}{2^9}$.

$Pr(\wedge \bar{A}_m)$ is the probability that A never occurs over a given m. In our case, $Pr(\wedge \bar{A}_m)$ is the probability that no 3x3 block is empty of a king. We let $m = nk$, so $Pr(\wedge \bar{A}_m)$ is the probability that every 3x3 block on an k by n board contains a king. In other words, $Pr(\wedge \bar{A}_m)$ is the probability of a dominating configuration for a k by n board.

x_i refers to the probability that a given event will occur independent of other events. x_j (for j from 1 to m) refers to the probabilities of the m events that can affect x_i. In terms of our problem, x_i is the probability that a given 3x3 block contains no kings. x_j ranges from 1 to 24, and is the probability that there will not be a king in each of the 24 surrounding 3x3 blocks. Since the probability of there being no king in a 3x3 block is always the same, $x_i = x_j = \frac{1}{2^9}$ for all j in our problem.

4.2 An Upper Bound for η

We can get an upper bound for η using the Local Lemma if we assume that $Pr(A_i)$ is completely independent of all other 3x3 blocks on a chessboard.

$$Pr(\wedge \bar{A}_m) \geq \prod_{i=1}^{m} (1 - x_i) \geq \eta$$

Because x_i remains constant in our problem ($x_i = \frac{1}{2^9}$), the middle product can be rewritten as $(1 - x_i)^m$. Now, from the above inequality we can generate another:

$$f(k, n) \geq \left(1 - \frac{1}{2^9}\right)^m * 2^m$$

We multiply the probability of getting a dominating configuration by the total number of ways to fill a k by n box to give us the actual number of dominating configurations. Some algebra on the right hand side of this inequality reveals:

$$\begin{aligned}\left(1-\frac{1}{2^9}\right)^m * 2^m &= \left(\frac{2^9-1}{2^9}\right)^m * 2^m \\ &= \left(\frac{2^9-1}{2^8}\right)^m \\ &= \left(2-\frac{1}{2^8}\right)^m\end{aligned}$$

Therefore,

$$f(k,n) \geq \left(2-\frac{1}{2^8}\right)^m$$

Recall that we calculate η by taking $f(n,k)^{\frac{1}{nk}}$. Since $m = nk$, we can raise both sides of the above inequality to the $\frac{1}{nk}$ and the inequality gives us our upper bound:

$$\eta \leq \left(2-\frac{1}{2^8}\right) \approx 1.99609375$$

4.3 A Lower Bound for η

To calculate a lower bound for η, we will look at the other equation in Lovasz Local Lemma:

$$Pr(A_i) \leq x_i \prod_{j}^{m} (1-x_j)$$

Recall that in our problem, $x_i = x_j$ for all j. So, the inequality becomes:

$$Pr(A_i) \leq x_i(1-x_i)^m$$

So for our problem, in order to get our lower bound we want to find the smallest possible x_i such that:

$$\frac{1}{2^9} \leq x_i(1-x_i)^{24}$$

Our upper bound relies on $\left(1-\frac{1}{2^9}\right)^m * 2^m$, where x_i is $\frac{1}{2^9}$, and we want our bounds to be as close as possible. Therefore, we want to get x_i for the lower bound slightly less than $\frac{1}{2^8}$, since:

$$\frac{1}{2^9} < \frac{1}{2^8}$$

which would make

$$\left(1-\frac{1}{2^9}\right)^m * 2^m > \left(1-\frac{1}{2^8}\right)^m * 2^m$$

To find such a value for x_i, we defined a function f:

$$f = x(1-x)^{24} - \frac{1}{2^9}$$

and solved for the smallest values of x that gave us a positive output.

Using Matlab, we were able to get x as small as .5252685547. Plugging this value of x back into our inequality from the Local Lemma, we get:

$$\begin{aligned} Pr(A_i) &\leq (1-x_i)^m \\ &\leq (1-.5252685547)^m \end{aligned}$$

As before, we multiply both sides of the inequality (the probabilities of dominating configurations) by the total possible number of configurations to get the actual number of dominating configurations:

$$f(k,n) \leq (1-.5252685547)^m * 2^m = (2-2(.5252685547))^m$$

Again, η comes from $f(k,n)^{\frac{1}{nk}}$ and $m = nk$. By taking both sides of the inequality to the $\frac{1}{nk}$ power, we get our lower bound:

$$\eta \geq (2-2(.5252685547)) \approx 1.995896367$$

5 Conclusions

There is still a great deal to learn about dominating configurations of kings on a chessboard. In the future, we would like to investigate the eigenvalues of different powers of adjacency matrices. In particular, we think the Perron-Frobenius theorem could be applied because the adjacency matrices are square, regular, and non-negative. Also, more work on the recursive definition of A_k could be done, including investigations similar to those done on the recursive definition for the knights problem. In addition, the bounds for both η and δ could be improved upon.

6 Acknowledgments

References

[1] S. Ahlgren and J. Lovejoy, *The arithmetic of partitions into distinct parts*, Mathematika **48** (2001), 203 - 211.