

FINITE FIELD ELEMENTS OF HIGH ORDER ARISING FROM MODULAR CURVES

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ABSTRACT. In this paper, we recursively construct explicit elements of provably high order in finite fields. We do this using the recursive formulas developed by Elkies to describe explicit modular towers. In particular, we give two explicit constructions based on two examples of his formulas and demonstrate that the resulting elements have high order. Between the two constructions, we are able to generate high order elements in every characteristic. Despite the use of the modular recursions of Elkies, our methods are quite elementary and require no knowledge of modular curves.

1. INTRODUCTION

Finding large order elements of finite fields has long been a problem of interest, particularly to cryptographers. Given a finite field \mathbb{F}_q , Gao [5] gives an algorithm for constructing elements of \mathbb{F}_{q^n} of order greater than

$$n^{\frac{\log_q n}{4 \log_q(2 \log_q n)} - \frac{1}{2}}.$$

The algorithm is nice because it makes no restriction on q and allows one to produce a provably high order element in any desired extension of \mathbb{F}_q provided that one can find a polynomial in $\mathbb{F}_q[x]$ with certain desirable properties. Gao conjectures that for any $n > 1$, there exists a polynomial of degree at most $2 \log_q n$ satisfying the conditions of his theorem. However, this conjecture remains unproven.

For special finite fields, it is possible to construct elements which can be proved to have much higher orders. See [6, 7, 10] on orders of Gauss periods and [2, 3] on Kummer extensions.

In [12], Voloch shows that under certain conditions, one of the coordinates of a point on a plane curve must have high order. The bounds we obtain through our methods have order of magnitude similar to those predicted in the main theorem of [12]. In a special case however, Voloch is able to achieve bounds which are much better. See section 5 of [12]. Unfortunately, Voloch does not fully state this theorem and only briefly mentions how one may adapt the proof of his main theorem for this special case. The bounds given in [12] are not as explicit as the ones we give here. Moreover, Voloch gives no explicit examples of his theorems. In Section 7 of this paper, we apply Voloch's technique to obtain a more explicit version of the special case of his main theorem.

In this paper, we consider elements in finite field towers recursively generated according to the equations for explicit modular towers [4]. We give two explicit constructions: one for odd characteristic and one for characteristic not equal to 3. In the first case, we explicitly

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construct elements of $\mathbb{F}_{q^{2^n}}$ whose orders are bounded below by $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$. In the second, we obtain elements of $\mathbb{F}_{q^{3^n}}$ whose orders are bounded below by $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$. Throughout we use the convention that exponentiation is right-associative, i.e., $a^{b^c} := a^{(b^c)}$.

2. CONSTRUCTIONS ARISING FROM MODULAR TOWERS

In [4], Elkies gives a recursive formula for the defining equations of the modular curve $X_0(\ell^n)$ by identifying $X_0(\ell^n)$ within the product $(X_0(\ell^2))^{n-1}$ for $n > 1$. For several cases, he even writes explicit equations. For example, in the case $\ell = 2$, the recursion is governed by the rule

$$(x_j^2 - 1) \left(\left(\frac{x_{j+1} + 3}{x_{j+1} - 1} \right)^2 - 1 \right) = 1 \text{ for } j = 1, 2, \dots, n - 2. \quad (1)$$

Elkies also notices that under a suitable change of variables and a reduction modulo 3, the equation becomes

$$y_{j+1}^2 = y_j - y_j^2,$$

which was used by Garcia and Stichtenoth [9] to recursively construct an asymptotically optimal function field tower. In fact, Elkies notes that many recursively constructed optimal towers may now be seen as arising from these modular curve constructions and speculates that perhaps all such towers are modular in this sense.

In this paper, we use Elkies' formulas to generate high order elements in towers of finite fields. For example, the following construction will yield high order elements in odd characteristic. The equation (1) may be manipulated to the form $f(X, Y) = 0$, where

$$f(X, Y) := Y^2 + (6 - 8X^2)Y + (9 - 8X^2), \quad (2)$$

and we have made the substitution $X = x_j$ and $Y = x_{j+1}$. Now, choose $q = p^m$ to be an odd prime power such that \mathbb{F}_q contains the fourth roots of unity (i.e. $q \equiv 1 \pmod{4}$). Choose $\alpha_0 \in \mathbb{F}_q$ such that $\alpha_0^2 - 1$ is not a square in \mathbb{F}_q . In Lemma 3 (see Section 3), we will show that such an α_0 always exists. Finally, define α_n by $f(\alpha_{n-1}, \alpha_n) = 0$ for $n \geq 1$. This construction yields the following result; where, as usual, for a prime ℓ , $\text{ord}_\ell(a)$ denotes the highest power of ℓ dividing a .

Theorem 1. *Let $\delta_n := \alpha_n^2 - 1$. Then δ_n has degree 2^n over \mathbb{F}_q , and the order of δ_n in $\mathbb{F}_{q^{2^n}}$ is greater than $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1)}$ unless $q \equiv 2 \pmod{3}$ and $\alpha_0 = \pm \left(\frac{p-1}{2}\right)$, in which case the order of δ_n is greater than $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$.*

To accommodate even characteristic, we have also considered Elkies' formula for $X_0(3^n)$. We will prefer to work with the equation in the polynomial form $g(X, Y) = 0$, where

$$g(X, Y) := Y^3 + (6 - 9X^3)Y^2 + (12 - 9X^3)Y + (8 - 9X^3). \quad (3)$$

For this construction, choose q to be a prime power congruent to 1 modulo 3 but not equal to 4. The condition $q \equiv 1 \pmod{3}$ assures the presence of the third roots of unity in \mathbb{F}_q . Choose $\beta_0 \in \mathbb{F}_q$ such that $\beta_0^3 - 1$ is not a cube in \mathbb{F}_q . In Lemma 4 (see Section 3), we show that such a β_0 always exists except when $q = 4$. Finally, define β_n by $g(\beta_{n-1}, \beta_n) = 0$ for $n \geq 1$. For this construction, we have the following result.

Theorem 2. *Let $\gamma_n := \beta_n^3 - 1$. Then γ_n has degree 3^n over \mathbb{F}_q , and the order of γ_n in $\mathbb{F}_{q^{3^n}}$ is greater than $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$.*

There are two interesting things about the above constructions. The first is that, computationally, the elements δ_n and γ_n appear to have much higher order than our bounds suggest. See Section 6 for examples. The second interesting thing is that, as with the case of the optimal function field tower constructions of Garcia and Stichtenoth [8, 9] arising from these modular curve recipes, our proofs do not at all exploit this modularity. Perhaps the key to achieving better bounds lies in this relationship.

The paper is organized as follows. In Section 3, we will state and prove some elementary number theory facts that will be of use to us. In Section 4, we consider the first tower; and in Section 5, we consider the second. Finally, in Section 6, we give a few examples of each of the main theorems.

3. NUMBER THEORETIC FACTS

Recall the following well known fact for detecting perfect n -th powers in finite fields. See [11, p. 81] for example.

Fact 1. *If $q \equiv 1 \pmod{n}$, then $x \in \mathbb{F}_q^*$ is a perfect n -th power if and only if $x^{(q-1)/n} = 1$.*

Also recall the following facts, which can be easily proved.

Fact 2. *Let $x \in \mathbb{F}_q^*$ of multiplicative order d . For $m, n \in \mathbb{N}$, if $x^n \neq 1$ and $x^{nm} = 1$, then $\gcd(d, m) > 1$.*

Fact 3. *Let $x \in \mathbb{F}_q^*$ of multiplicative order d . If ℓ is a prime, $m = \text{ord}_\ell(n)$, and x^n is a nontrivial ℓ -th root of unity, then ℓ^{m+1} divides d .*

The following lemmas are useful for bounding the orders of the elements appearing in Theorems 1 and 2.

Lemma 1. *Let $\ell, b \in \mathbb{N}$ such that $b \equiv 1 \pmod{\ell}$, and let $M, N \in \mathbb{N}$ with $M < N$. Then*

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}, \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}\right) = \ell;$$

and hence $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$ and $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$ are coprime.

Proof. The following computation follows from Euclid's algorithm:

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}, b^{\ell^N} - 1\right) = \gcd\left(\ell, b^{\ell^N} - 1\right) = \ell. \quad (4)$$

Since $M < N$, repeatedly using the difference of ℓ -th powers formula shows that $\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$ divides $b^{\ell^N} - 1$. Also, since $b \equiv 1 \pmod{\ell}$, it is clear that ℓ divides both $\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$ and $\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$. Therefore,

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}, \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}\right) = \ell.$$

□

Lemma 2. *Let $\ell, b, N \in \mathbb{N}$ with ℓ prime and $b \equiv 1 \pmod{\ell}$. If p is a prime dividing $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$, then $p > \ell^{N+1}$.*

Proof. Since $\ell \geq 2$ and $b \equiv 1 \pmod{\ell}$, ℓ^2 divides $(b^{\ell^N} - 1)$. Hence, $p \neq \ell$ for otherwise, we have a contradiction with (4). Thus, p dividing $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$ implies that $\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)} \equiv 0 \pmod{p}$. So, b^{ℓ^N} is a nontrivial ℓ -th root of unity modulo p . Therefore, by Fact 3, ℓ^{N+1} divides $p - 1$, and hence $p > \ell^{N+1}$. \square

The following two lemmas essentially give the necessary and sufficient conditions for completing the first step in the construction of our towers, i.e., under certain restrictions on q , they demonstrate the existence of α_0 and β_0 each having its desired property. The proofs involve counting \mathbb{F}_q solutions to equations via character sums. We refer the reader to [11, Chapter 8] for more on this technique. As in [11], for characters ψ and λ on \mathbb{F}_q , we denote the Jacobi sum of ψ and λ by $J(\psi, \lambda) := \sum_{a+b=1} \psi(a)\lambda(b)$.

Lemma 3. *Let q be a prime power. Then there exists $\alpha_0 \in \mathbb{F}_q$ such that $\delta_0 = \alpha_0^2 - 1$ is not a square in \mathbb{F}_q if and only if q is odd and $q \neq 3$.*

Proof. First, note that if q is even, then every element of \mathbb{F}_q is a square. So, we assume that q is odd. We desire $\alpha_0 \in \mathbb{F}_q^*$ such that $\alpha_0^2 - 1$ is not a square. Our method for proving that such an α_0 exists involves counting solutions to the equation $x^2 - y^2 = 1$. Let τ be the unique character of exact order 2 on \mathbb{F}_q . Then

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^2 - y^2 = 1\} &= \sum_{\substack{a, b \in \mathbb{F}_q \\ a+b=1}} \left(\sum_{j=0}^1 \tau^j(a) \right) \left(\sum_{j=0}^1 \tau^j(-b) \right) \\ &= q + \tau(-1)J(\tau, \tau) = q - 1. \end{aligned}$$

On the other hand, if $\alpha_0^2 - 1$ is a square for all choices of α_0 , then $\alpha_0^2 - 1 = y^2$ has a solution for all $\alpha_0 \in \mathbb{F}_q$. In this case, we have

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^2 - y^2 = 1\} &= \sum_{\alpha_0 \in \mathbb{F}_q} \#\{y \in \mathbb{F}_q : y^2 = \alpha_0^2 - 1\} \\ &= 2 + \#\{y^2 = -1\} + \sum_{\substack{\alpha_0 \in \mathbb{F}_q \\ \alpha_0 \neq 0, \pm 1}} \#\{y^2 = \alpha_0^2 - 1\} \\ &= 2q - 3 + \tau(-1). \end{aligned}$$

Thus, the assumption that $\alpha_0^2 - 1$ is always a square leads to the conclusion $q = 2 - \tau(-1)$, which implies $q = 3$. \square

The requirement that $q \neq 3$ will pose no problem since we assume that $q \equiv 1 \pmod{4}$ for the construction of the tower of Theorem 1.

Lemma 4. *Let q be a prime power. Then there exists $\beta_0 \in \mathbb{F}_q$ such that $\gamma_0 = \beta_0^3 - 1$ is not a cube in \mathbb{F}_q if and only if $q \equiv 1 \pmod{3}$ and $q \neq 4$.*

Proof. First, note that if $q \not\equiv 1 \pmod{3}$, then every element of \mathbb{F}_q is a cube. So, we will assume that $q \equiv 1 \pmod{3}$. As mentioned earlier, this means that \mathbb{F}_q contains a primitive third root of unity. We now count \mathbb{F}_q solutions to the equation $x^3 - y^3 = 1$. Let χ be any character of order 3 on \mathbb{F}_q .

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^3 - y^3 = 1\} &= \sum_{\substack{a, b \in \mathbb{F}_q, \\ a+b=1}} \left(\sum_{j=0}^2 \chi^j(a) \right) \left(\sum_{j=0}^2 \chi^j(-b) \right) = \sum_{i=0}^2 \sum_{j=0}^2 \chi^j(-1) J(\chi^i, \chi^j) \\ &= q - 2\chi(-1) + J(\chi, \chi) + J(\chi^2, \chi^2) = q - 2 + 2\operatorname{Re}J(\chi, \chi). \end{aligned}$$

On the other hand, if we assume that $\beta_0^3 - 1$ is a cube for all choices of $\beta_0 \in \mathbb{F}_q$, then

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^3 - y^3 = 1\} &= \sum_{\beta_0 \in \mathbb{F}_q} \#\{y \in \mathbb{F}_q : \beta_0^3 - y^3 = 1\} \\ &= \sum_{\beta_0^3=1} 1 + \sum_{\beta_0^3 \neq 1} 3 = 3 + 3(q-3) = 3q - 6. \end{aligned}$$

Thus, the assumption that $\beta_0^3 - 1$ is always a cube leads to the conclusion that $|2q - 4| = |(3q - 6) - (q - 2)| = |2\operatorname{Re}J(\chi, \chi)| \leq 2\sqrt{q}$, which implies $|q - 2| \leq \sqrt{q}$. This implies that $(q - 1)(q - 4) \leq 0$. The only $q \equiv 1 \pmod{3}$ satisfying this inequality is $q = 4$. \square

4. THE QUADRATIC TOWER FOR ODD CHARACTERISTIC

In this section, we consider the first tower, which is recursively constructed using (2). Throughout this section we will assume that p is an odd prime and that $q = p^m \equiv 1 \pmod{4}$. In particular, if $p \equiv 3 \pmod{4}$, then $2|m$. As discussed in the introduction, this condition ensures the existence of a primitive fourth root of unity. This will be seen to be a necessary ingredient in the construction of our tower. We also fix α_0 such that $\delta_0 = \alpha_0^2 - 1$ is not a square in \mathbb{F}_q . Recall that Lemma 3 ensures the existence of such an α_0 .

Before moving forward, we need to establish the relationship between δ_n and δ_{n-1} . From (2) and the definition of δ_n (see Theorem 1), we deduce that δ_{n-1} and δ_n are related by $F(\delta_{n-1}, \delta_n) = 0$ ($n \geq 1$), where

$$F(X, Y) := Y^2 - (48X + 64X^2)Y - 64X. \quad (5)$$

We also fix the following more compact notation for the norm. We take

$$\begin{aligned} N_{n,j} : \mathbb{F}_{q^{2^n}} &\rightarrow \mathbb{F}_{q^{2^{n-j}}}, \\ \alpha &\mapsto \alpha^{\prod_{k=1}^j (q^{2^{n-k}} + 1)}. \end{aligned}$$

For the purpose of making the proof easier to digest, we break Theorem 1 into a pair of propositions.

Proposition 1. *The elements α_n and δ_n have degree 2 over $\mathbb{F}_{q^{2^{n-1}}}$ for $n \geq 1$.*

Proof. First note that the discriminant of $f(\alpha_{n-1}, Y)$ is $\delta_{n-1} = \alpha_{n-1}^2 - 1$ for all $n \geq 1$. We will proceed by induction on n . Recall that α_0 was chosen so that δ_0 , the discriminant of $f(\alpha_0, Y)$, is not a square in \mathbb{F}_q . Thus, α_1 satisfies an irreducible polynomial of degree 2 over \mathbb{F}_q , i.e., α_1 has degree 2 over \mathbb{F}_q . We may take $\{1, \alpha_1\}$ as a basis for $\mathbb{F}_q(\alpha_1)$ over \mathbb{F}_q . Writing δ_1 in terms of the basis, we have $\delta_1 = \alpha_1^2 - 1 = (8\alpha_0^2 - 6)\alpha_1 + (8\alpha_0^2 - 10)$. So, $\delta_1 \in \mathbb{F}_q$ if and

only if $8\alpha_0^2 - 6 = 0$. If $8\alpha_0^2 - 6 = 0$, then $\delta_0 = \alpha_0^2 - 1 = -4^{-1}$, which is a square in \mathbb{F}_q since \mathbb{F}_q contains the fourth roots of unity. This is contrary to our choice of α_0 . Thus, δ_1 has degree 2 over \mathbb{F}_q as well.

Now, suppose that α_k and δ_k both have degree 2 over $\mathbb{F}_{q^{2^{k-1}}}$ for $1 \leq k \leq n$. Then $f(\alpha_{n-1}, Y)$ is the minimum polynomial of α_n over $\mathbb{F}_{q^{2^{n-1}}}$; and hence, the discriminant is not a square in $\mathbb{F}_{q^{2^{n-1}}}$. In particular,

$$\delta_{n-1}^{(q^{2^{n-1}}-1)/2} = -1. \quad (6)$$

We also see that $F(\delta_{n-1}, Y)$ is the minimum polynomial of δ_n over $\mathbb{F}_{q^{2^{n-1}}}$. To see that the degree of α_{n+1} over $\mathbb{F}_{q^{2^n}}$ is 2, we show that $f(\alpha_n, Y)$ is irreducible over $\mathbb{F}_{q^{2^n}}$. Now,

$$\begin{aligned} \delta_n^{(q^{2^n}-1)/2} &= \left(\delta_n^{(q^{2^{n-1}}+1)} \right)^{(q^{2^{n-1}}-1)/2} = (N_{n,1}(\delta_n))^{(q^{2^{n-1}}-1)/2} \\ &= (-64\delta_{n-1})^{(q^{2^{n-1}}-1)/2} = -1. \end{aligned}$$

Here we have used (6) and the fact that -64 is a square in $\mathbb{F}_{q^{2^{n-1}}}$ since \mathbb{F}_q contains the fourth roots of unity. Thus, δ_n is not a square, and hence $f(\alpha_n, Y)$ is irreducible. So, the set $\{1, \alpha_{n+1}\}$ forms a basis for $\mathbb{F}_{q^{2^{n+1}}}$ over $\mathbb{F}_{q^{2^n}}$. Now, we write δ_{n+1} in terms of the basis, and apply the same argument as for δ_1 to see that the degree of δ_{n+1} over $\mathbb{F}_{q^{2^n}}$ is 2 as well. This completes the induction and the proof. \square

An easy induction proof, exploiting the fact that $F(\delta_{k-1}, Y)$ is the minimum polynomial of δ_k over $\mathbb{F}_{q^{2^{k-1}}}$ for $1 \leq k \leq n$, shows that

$$N_{n,j}(\delta_n) = (-64)^{(2^j-1)} \delta_{n-j} \quad (7)$$

for $1 \leq j \leq n$. This fact will be useful in the proof of the proposition below.

Proposition 2. *The order of δ_n in $\mathbb{F}_{q^{2^n}}$ is greater than $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1)}$ unless $q \equiv 2 \pmod{3}$ and $\alpha_0 = \pm \left(\frac{p-1}{2}\right)$, in which case the order of δ_n is greater than $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$.*

Proof. We first compute the power of 2 dividing the order of δ_n . Recall from the proof of Proposition 1 that $\delta_n^{(q^{2^n}-1)/2} \neq 1$; but of course, $\delta_n^{(q^{2^n}-1)} = 1$ since $\delta_n \in \mathbb{F}_{q^{2^n}}$. Since $q \equiv 1 \pmod{4}$, $\text{ord}_2(q^{2^j} + 1) = 1$ for each $j \geq 1$. Repeatedly using the difference of squares formula, we have

$$\begin{aligned} \text{ord}_2\left(\frac{q^{2^n} - 1}{2}\right) &= \text{ord}_2(q - 1) - 1 + \sum_{j=0}^{n-1} \text{ord}_2(q^{2^j} + 1) \\ &= n - 1 + \text{ord}_2(q - 1). \end{aligned}$$

Thus, $2^{n+\text{ord}_2(q-1)}$ divides the order of δ_n by Fact 3.

Now we look for odd primes dividing the order. By Fact 2, the order of δ_n has a common factor with $(q^{2^{n-j}} + 1)/2$ for each j such that

$$\frac{(q^{2^n} - 1)}{\delta_n^{(q^{2^{n-j}} + 1)/2}} \neq 1.$$

By (7), we have

$$\begin{aligned} \frac{(q^{2^n}-1)}{\delta_n^{(q^{2^{n-j}}+1)/2}} &= (\mathbb{N}_{n,j-1}(\delta_n))^{2(q^{2^{n-j}}-1)} = ((-64)^{(2^{j-1}-1)}\delta_{n-j+1})^{2(q^{2^{n-j}}-1)} \\ &= (\delta_{n-j+1})^{2(q^{2^{n-j}}-1)} \neq 1 \end{aligned}$$

provided that $\delta_{n-j+1}^2 \notin \mathbb{F}_{q^{2^{n-j}}}$. From (5), we know that we may write δ_{n-j+1}^2 as

$$\delta_{n-j+1}^2 = (48\delta_{n-j} + 64\delta_{n-j}^2)\delta_{n-j+1} + 64\delta_{n-j}.$$

Thus, $\delta_{n-j+1}^2 \in \mathbb{F}_{q^{n-j}}$ if and only if δ_{n-j} satisfies the equation $48\delta_{n-j} + 64\delta_{n-j}^2 = 0$. If this were the case, then $\delta_{n-j} = 0$ or $\delta_{n-j} = -3^{-1}4$. By Proposition 1, this implies that $n = j$. However, $\delta_0 = 0$ contradicts the choice of α_0 ; and $\delta_0 = -4^{-1}3$ contradicts the choice of α_0 unless -3 is not a perfect square, that is, unless $q \equiv 2 \pmod{3}$. If $q \equiv 2 \pmod{3}$, then the only choices of α_0 that give $\delta_0 = -4^{-1}3$ are $\alpha_0 = \pm \left(\frac{p-1}{2}\right)$. Thus, the order of δ_n has a common factor with $(q^{2^{n-j}} + 1)/2$ for each $1 \leq j \leq n$ unless $q \equiv 2 \pmod{3}$, $\alpha_0 = \pm \left(\frac{p-1}{2}\right)$, and $j = n$. Each of these factors must be odd since $\text{ord}_2(q^{2^{n-j}} + 1) = 1$ as noted above. By Lemma 1 with $\ell = 2$ and $b = q$, we see that these factors must be pairwise coprime as well. Hence, we get either n or $n - 1$ distinct odd prime factors dividing the order of δ_n depending on the case. By Lemma 2, we see that each such prime factor must be bounded below by 2^{n-j+1} . Therefore, the order of δ_n is bounded below by

$$2^{n+\text{ord}_2(q-1)} \prod_{j=1}^n 2^{n-j+1} = 2^{n+\text{ord}_2(q-1)+n(n+1)/2} = 2^{\frac{n^2+3n}{2}+\text{ord}_2(q-1)}$$

unless $q \equiv 2 \pmod{3}$ and $\alpha_0 = \pm \left(\frac{p-1}{2}\right)$, in which case the order is bounded below by $2^{\frac{1}{2}n^2+\frac{3}{2}n+\text{ord}_2(q-1)-1}$. \square

Theorem 1 follows by combining the two propositions. The authors would like to point out that it is possible to achieve a slightly better lower bound for the order of δ_n by the following method. First, choose a square root of δ_{n-1} , say $\sqrt{\delta_{n-1}} \in \mathbb{F}_{q^{2^n}}$. Then use the method above to prove a lower bound for the order of $\sqrt{\delta_{n-1}}$. Finally, deduce a bound for the order of δ_n . The improvement, however, only affects the coefficient of n in the exponent. Since computationally our bounds do not appear to be that close to the truth, we have decided to work directly with δ_n instead.

5. THE CUBIC TOWER FOR CHARACTERISTIC NOT 3

In this section, we consider the second tower, which is recursively constructed using (3). Recall that, for this tower, we assume that $q \equiv 1 \pmod{3}$ and $q \neq 4$. This means that \mathbb{F}_q will contain the third roots of unity, and hence the third roots of -1 as well. We also fix a β_0 such that $\gamma_0 = \beta_0^3 - 1$ is not a cube in \mathbb{F}_q . Recall that Lemma 4 ensures the existence of such a β_0 .

Before we begin the proof of Theorem 2, we need to establish the relationship between γ_{n-1} and γ_n . The relationship is given by $G(\gamma_{n-1}, \gamma_n) = 0$ for $n \geq 1$, where

$$G(X, Y) := Y^3 - (270X + 972X^2 + 729X^3)Y^2 - (972X + 729X^2)Y - 729X. \quad (8)$$

This follows from (3) and the definition of γ_n . We also fix the following notation for the norm.

$$\begin{aligned} N_{n,j} : \mathbb{F}_{q^{3^n}} &\rightarrow \mathbb{F}_{q^{3^{n-j}}}, \\ \beta &\mapsto \beta^{\prod_{k=1}^j \left((q^{3^{n-k}})^2 + q^{3^{n-k}} + 1 \right)}. \end{aligned}$$

As in section 4, we break the result into two smaller propositions.

Proposition 3. *The elements β_n and γ_n both have degree 3 over $\mathbb{F}_{q^{3^{n-1}}}$ for $n \geq 1$.*

Proof. By carefully examining the cubic formula applied to the polynomial, one can see that $g(\beta_{n-1}, Y)$ is irreducible if and only if $\gamma_{n-1} = \beta_{n-1}^3 - 1$ is not a cube in $\mathbb{F}_{q^{3^{n-1}}}$. Thus, β_n will have degree 3 over $\mathbb{F}_{q^{3^{n-1}}}$ if and only if γ_{n-1} is not a cube in $\mathbb{F}_{q^{3^{n-1}}}$ for all $n \geq 1$. As with the proof of Proposition 1, we proceed by induction on n . Recall that β_0 was chosen so that γ_0 is not a cube in \mathbb{F}_q . Thus, β_1 has degree 3 over \mathbb{F}_q . So, we may take $\{1, \beta_1, \beta_1^2\}$ as a basis for \mathbb{F}_{q^3} over \mathbb{F}_q . Writing γ_1 in terms of the basis, we have

$$\gamma_1 = \beta_1^3 - 1 = (9\beta_0^3 - 6)\beta_1^2 + (9\beta_0^3 - 12)\beta_1 + (9\beta_0^3 - 9).$$

So, $\gamma_1 \in \mathbb{F}_q$ if and only if $9\beta_0^3 - 6 = 0$ and $9\beta_0^3 - 12 = 0$. This leads to the conclusion that $\gamma_0 = -3^{-1}$ and $\gamma_0 = 3^{-1}$, which implies that $2 = 0$, i.e., the characteristic is 2. In this case, we are led to the conclusion that $\gamma_0 = 1$, which is a cube. This of course is contrary to our choice of γ_0 . Therefore, $\gamma_1 \notin \mathbb{F}_q$, i.e., the degree of γ_1 over \mathbb{F}_q is 3. This completes the trivial case.

Now, let ω be a primitive cube root of unity in \mathbb{F}_q and suppose that β_k and γ_k both have degree 3 over $\mathbb{F}_{q^{3^{k-1}}}$ for $1 \leq k \leq n$. Then $g(\beta_{n-1}, Y)$ is the minimum polynomial of β_n over $\mathbb{F}_{q^{3^{n-1}}}$; and hence γ_{n-1} is not a cube in $\mathbb{F}_{q^{3^{n-1}}}$. In particular,

$$\gamma_{n-1}^{(q^{3^{n-1}}-1)/3} = \omega.$$

We also see that $G(\gamma_{n-1}, Y)$ is the minimum polynomial of γ_n over $\mathbb{F}_{q^{3^{n-1}}}$. Thus,

$$\begin{aligned} \gamma_n^{(q^{3^n}-1)/3} &= \left(\gamma_n^{\left((q^{3^{n-1}})^2 + q^{3^{n-1}} + 1 \right)} \right)^{(q^{3^{n-1}}-1)/3} = (N_{n,1}(\gamma_n))^{(q^{3^{n-1}}-1)/3} \\ &= (-729\gamma_{n-1})^{(q^{3^{n-1}}-1)/3} = \omega; \end{aligned}$$

i.e., β_{n+1} has degree 3 over $\mathbb{F}_{q^{3^n}}$. To see that γ_{n+1} also has degree 3 over $\mathbb{F}_{q^{3^n}}$, write γ_{n+1} in terms of the $\mathbb{F}_{q^{3^n}}$ -basis $\{1, \beta_{n+1}, \beta_{n+1}^2\}$, and proceed as we did for γ_1 . \square

An easy induction proof using the fact that $G(\gamma_{k-1}, Y)$ is the minimum polynomial of γ_k over $\mathbb{F}_{q^{3^{k-1}}}$ for $1 \leq k \leq n$, shows that

$$N_{n,j}(\gamma_n) = (-729)^{(3^j-1)} \gamma_{n-j}$$

for $1 \leq j \leq n$.

Proposition 4. *The order of γ_n in $\mathbb{F}_{q^{3^n}}$ is greater than $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$.*

Proof. We first compute the power of 3 dividing the order of γ_n . Recall from the proof of Proposition 3 that $\gamma_n^{(q^{3^n}-1)/3} \neq 1$. However, $\gamma_n^{(q^{3^n}-1)} = 1$ since $\gamma_n \in \mathbb{F}_{q^{3^n}}$. Since $q \equiv 1 \pmod{3}$, $\text{ord}_3((q^{3^j})^2 + q^{3^j} + 1) = 1$ for each $j \geq 1$. Repeatedly using the difference of cubes formula, we have

$$\begin{aligned} \text{ord}_3\left(\frac{q^{3^n}-1}{3}\right) &= \text{ord}_3(q-1) - 1 + \sum_{j=0}^{n-1} \text{ord}_3\left(\left(q^{3^j}\right)^2 + q^{3^j} + 1\right) \\ &= n - 1 + \text{ord}_3(q-1). \end{aligned}$$

Thus, $3^{n+\text{ord}_3(q-1)}$ divides the order of γ by Fact 3.

Now, we look for primes dividing the order that are not equal to 3. In particular, we will show that the order of γ_n has a common factor with $((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3$ for each $1 \leq j \leq n$. This factor must not be a multiple of 3 since $\text{ord}_3((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1) = 1$ as noted above. By Lemma 1, with $\ell = 3$ and $b = q$, we see that these factors must be pairwise coprime as well. Hence, we get n distinct prime factors dividing the order of γ_n , none of which are equal to 3. By Lemma 2, each of these primes must be bounded below by 3^{n-j+1} . Hence, if we can show that the order of γ_n has a common factor with $((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3$ for $1 \leq j \leq n$, then we have that the order of γ_n is bounded below by

$$3^{n+\text{ord}_3(q-1)} \prod_{j=1}^n 3^{n-j+1} = 3^{n+\text{ord}_3(q-1)+n(n+1)/2} = 3^{\frac{n^2+3n}{2}+\text{ord}_3(q-1)}.$$

By Fact 2, the proof will be complete when we show that

$$\frac{q^{3^n}-1}{((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3} \neq 1 \text{ for } 1 \leq j \leq n.$$

Now,

$$\frac{q^{3^n}-1}{((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3} = (\mathbb{N}_{n,j-1}(\gamma_n))^{3(q^{3^{n-j}}-1)} = ((-729)^{(3^{j-1}-1)}\gamma_{n-j+1})^{3(q^{3^{n-j}}-1)} \neq 1$$

provided $\gamma_{n-j+1}^3 \notin \mathbb{F}_{q^{3^{n-j}}}$. From (8), we know that we may write γ_{n-j+1}^3 as

$$\gamma_{n-j+1}^3 = (270\gamma_{n-j} + 972\gamma_{n-j}^2 + 729\gamma_{n-j}^3)\gamma_{n-j+1}^2 + (972\gamma_{n-j} + 729\gamma_{n-j}^2)\gamma_{n-j+1} + 729\gamma_{n-j}.$$

Thus, $\gamma_{n-j+1}^3 \in \mathbb{F}_{q^{3^{n-j}}}$ if only if γ_{n-j} satisfies the system

$$\begin{aligned} 270\gamma_{n-j} + 972\gamma_{n-j}^2 + 729\gamma_{n-j}^3 &= 0, \\ 972\gamma_{n-j} + 729\gamma_{n-j}^2 &= 0. \end{aligned}$$

Suppose that γ_{n-j} does satisfy the above system. If the characteristic is 2, the first equation implies that $\gamma_{n-j} = 0$, which is a contradiction. Suppose then that the characteristic is not 2. Solving the system, we see that $-3^{-2}(6 + \sqrt{6}) = \gamma_{n-j} = -3^{-1}4$, where $\sqrt{6}$ may be any square root of 6. This leads to the conclusion that $30 = 0$. Hence, the characteristic must be 5. By Proposition 3, we see that $j = n$ since $\gamma_{n-j} = -3^{-1}4 \in \mathbb{F}_q$. However, this means that $\gamma_0 = 2$, which is in contradiction with the choice of β_0 since 2 is a perfect cube in this case. \square

6. EXAMPLES OF THEOREMS

In this section we provide the data from the first several iterations for five examples of the main theorems: three for Theorem 1 and two for Theorem 2. The tables in this section provide information about the orders of α_n , β_n , δ_n , and γ_n in relation to our bound. We have chosen to take logs of these numbers because of their size. In each example, we see that the actual orders are much higher than our lower bounds. Computations were aided by MAGMA [1].

For our first example of Theorem 1, we choose $q = 5$ and $\alpha_0 = 2$.

TABLE 1. $q = 5$; $\alpha_0 = 2$.

n	$\log_2 \mathbb{F}_{5^{2n}}^* $	$\log_2 \langle \alpha_n \rangle $	$\log_2 \langle \delta_n \rangle $	$\log_2 \left(2^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	4.59	4.59	3.00	3.00
2	9.28	9.28	7.70	6.00
3	18.6	16.0	17.0	10.0
4	37.1	35.6	31.5	15.0
5	74.2	69.8	68.6	21.0
6	148.	148.	143.	28.0
7	297.	295.	292.	36.0
8	594.	590.	589.	45.0

For our second example of Theorem 1, we choose $q = 9$ and $\alpha_0 = \zeta + 2$, where ζ is a root of $x^2 + 1$. Note that, in this example, δ_n is actually primitive for each of the first eight iterations.

TABLE 2. $q = 9$; $\alpha_0 = \zeta + 2$.

n	$\log_2 \mathbb{F}_{9^{2n}}^* $	$\log_2 \langle \alpha_n \rangle $	$\log_2 \langle \delta_n \rangle $	$\log_2 \left(2^{\frac{1}{2}n^2 + \frac{3}{2}n + 3} \right)$
1	6.32	5.32	6.32	5.00
2	12.7	10.7	12.7	8.00
3	25.4	22.4	25.4	12.0
4	50.8	46.8	50.8	17.0
5	102.	96.5	102.	23.0
6	203.	197.	203.	30.0
7	406.	399.	406.	38.0
8	812.	804.	812.	47.0

For our final example of Theorem 1, we choose $q = 121$ and $\alpha_0 = \eta^8$, where η is a root of $x^2 + 7x + 2$. Here, δ_n is primitive except for $n = 3$ and $n = 7$.

TABLE 3. $q = 121$; $\alpha_0 = \eta^8$.

n	$\log_2 \mathbb{F}_{121^{2^n}}^* $	$\log_2 \langle \alpha_n \rangle $	$\log_2 \langle \delta_n \rangle $	$\log_2 \left(2^{\frac{1}{2}n^2 + \frac{3}{2}n + 3} \right)$
1	13.8	11.8	13.8	5.00
2	27.7	26.7	27.7	8.00
3	55.4	50.8	53.0	12.0
4	111.	109.	111.	17.0
5	222.	216.	222.	23.0
6	443.	440.	443.	30.0
7	886.	874.	883.	38.0

For our first example of Theorem 2, we choose $q = 7$ and $\beta_0 = 3$. In this example, γ_n appears to alternate between being primitive and not.

TABLE 4. $q = 7$; $\beta_0 = 3$.

n	$\log_2 \mathbb{F}_{7^{3^n}}^* $	$\log_2 \langle \beta_n \rangle $	$\log_2 \langle \gamma_n \rangle $	$\log_2 \left(3^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	8.42	7.41	5.84	4.76
2	25.3	25.3	25.3	9.52
3	75.8	75.8	74.2	15.8
4	228.	228.	228.	23.8
5	682.	681.	681.	33.3

For our second example of Theorem 2, we choose $q = 16$ and $\beta_0 = \xi$, where ξ is a root of $x^4 + x + 1$. Note that here γ_n is primitive for each of the first five iterations.

TABLE 5. $q = 16$; $\beta_0 = \xi$.

n	$\log_2 \mathbb{F}_{16^{3^n}}^* $	$\log_2 \langle \beta_n \rangle $	$\log_2 \langle \gamma_n \rangle $	$\log_2 \left(3^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	12.0	8.83	12.0	4.76
2	36.0	31.2	36.0	9.52
3	108.	102.	108.	15.8
4	324.	316.	324.	23.8
5	972.	962.	972.	33.3

7. COMPARISON WITH VOLOCH'S WORK

The following is an improvement of a result of Voloch [12, §5]. The proof is along the similar lines as that presented in [12] but more elementary in the sense that we avoid working with algebraic function fields in general.

Theorem 3. *Let $0 < \epsilon < 1/2$ be given, and suppose that r and d are integers such that $\gcd(r, q) = 1$, $r \leq d^{1-2\epsilon}$, $d^\epsilon \geq 3$ and $\lceil d^{1-\epsilon} \rceil < d$. Note that all of these conditions are satisfied for d sufficiently large. If $a \in \overline{\mathbb{F}}_q$ has degree d over \mathbb{F}_q and multiplicative order r then $a - 1$ has multiplicative order at least $\exp(\frac{\epsilon}{3}d^{\epsilon/3} \log d)$.*

Proof. Let $0 < \epsilon < 1/2$ be given, and put $N := \lceil d^{1-\epsilon} \rceil$. For each coset Γ of $\langle q \rangle$ in $(\mathbb{Z}/r\mathbb{Z})^*$, we define

$$J_\Gamma := \{n \leq N : n \pmod r \in \Gamma\}.$$

Then

$$\sum_{\Gamma} |J_\Gamma| = \#\{1 \leq n \leq N : \gcd(n, r) = 1\} \geq \left\lfloor \frac{N}{r} \right\rfloor \phi(r),$$

where the sum is over all cosets of Γ in $(\mathbb{Z}/r\mathbb{Z})^*$. Thus there exists a coset $\Gamma = \gamma\langle q \rangle$ such that $|J_\Gamma|$ is at least the average, i.e.,

$$|J_\Gamma| \geq \left\lfloor \frac{N}{r} \right\rfloor \frac{\phi(r)}{[(\mathbb{Z}/r\mathbb{Z})^* : \langle q \rangle]} \geq \left\lfloor \frac{N}{r} \right\rfloor \geq \lfloor d^\epsilon \rfloor.$$

Choose $a \in \overline{\mathbb{F}}_q$ such that $\deg_{\mathbb{F}_q} a = d$ and with multiplicative order r . Let $c \in \overline{\mathbb{F}}_q$ such that $a = c^\gamma$. Then $\deg_{\mathbb{F}_q} c \geq d$ and c also has multiplicative order r . Let $b = a - 1$.

For each $n \in J_\Gamma$, there exists j_n such that $n \equiv \gamma q^{j_n} \pmod r$. Whence $c^n = c^{\gamma q^{j_n}} = a^{q^{j_n}}$, and so $b^{q^{j_n}} = a^{q^{j_n}} - 1 = c^n - 1$.

Now, for every $I \subset J_\Gamma$ we write $b_I := \prod_{n \in I} (c^n - 1) = \prod_{n_j \in I} b^{q^{n_j}}$ which is a power of b .

Put $T = \lceil d^{\epsilon/3} \rceil$. Observe that $T < d/N$. We claim that for all distinct $I, I' \subset J_\Gamma$ with $|I| = |I'| = T$ we have that $b_I \neq b_{I'}$. Suppose that $b_I = b_{I'}$, and consider the non-zero polynomial

$$p(t) = \prod_{n \in I} (t^n - 1) - \prod_{n \in I'} (t^n - 1).$$

Then $p(c) = b_I - b_{I'} = 0$ so $\deg p(t) \geq \deg_{\mathbb{F}_q} c \geq d$. But on the other hand we have that $\deg p(t) \leq NT < d$, a contradiction. Thus $b_I \neq b_{I'}$ as claimed.

It follows that there are at least $\binom{|J_\Gamma|}{T}$ distinct powers of b . Now

$$\binom{|J_\Gamma|}{T} \geq \left(\frac{|J_\Gamma|}{T} - 1 \right)^T \geq (\lfloor d^\epsilon \rfloor / d^{\epsilon/3} - 1)^{d^{\epsilon/3}} \geq (d^{\epsilon/3})^{d^{\epsilon/3}} = \exp\left(\frac{\epsilon}{3}d^{\epsilon/3} \log d\right)$$

as required. □

In order to compare this result to Theorem 1, one may choose a to be a primitive 2^n -th root of unity in $\overline{\mathbb{F}}_q$. Then, for n sufficiently large, the conditions of the above theorem will be satisfied, and the degree of a over \mathbb{F}_q will be $2^{n - \text{ord}_2(q-1)}$. Similarly, one may choose a to be a primitive 3^n -th root of unity in $\overline{\mathbb{F}}_q$ to compare with Theorem 2.

Because of the the requirement that a must have low order relative to its degree, there are many fields where Theorem 3 will not apply. Furthermore, one may check that even though the bound of Theorem 3 will eventually dominate the bounds of Theorems 1 and 2, there will always be a range (in terms of n) in which bounds of Theorems 1 and 2 will be larger.

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