Cuts of the Hypercube

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Abstract

This paper approaches the hypercube slicing problem from a probabilistic perspective. We select a probability model and find several probabilities related to this problem, often using geometry to find simple expressions for complicated-looking integrals. Using these probabilities, we obtain several results, including the number of random hyperplanes required to cut all edges of the hypercube with high probability, a probability-preserving bijection between homogeneous and nonhomogeneous planes in neighboring dimensions, and an inter-dimensional relationship between the probabilities.

1 Introduction

Let d represent the dimension in which a hypercube exists. Also, let c represent the number of edges cut by a hyperplane. Does there exist four hyperplanes in $\mathbb{R}^5$ that cut every edge of a d-dimensional hypercube? In general, what is the smallest number of hyperplanes necessary in $\mathbb{R}^d$ so that c equals the number of edges in a hypercube of d-dimension? Patrick E. O'Neil originally proposes this problem [3]. With the usage of Baker's Generalization of Sperner's Lemma and Stirling's Formula, O'Neil proves a theorem which states that the maximal number of edges of a hypercube which may be cut by a Hyperplane is given by

$$k = \left(n - \left[\frac{1}{2}n\right]\right)\binom{n}{\left[\frac{n}{2}\right]} . \tag{1}$$

Sohler and Ziegler expand greatly on O'Neil's proposal [4] and are even able to provide a lower bound for the minimum number of hyperplanes needed to cut all edges of a hypercube in dimensions up to seven. They also gather enough evidence to prove that is takes only 5 hyperplanes to cut all edges of a 5^{th} dimensional hypercube, only 5 hyperplanes to cut all edges of a 6^{th} dimensional hypercube, and 5 or 6 hyperplanes to cut all edges of a 7^{th} dimensional hypercube. Michael Saks, another significant contributor the problem, poses further questions regarding generalizations of the problem and considers additional elements such as subhypercubes.

In order to attack this problem, we use probabilistic methods which reveal to us the likelihood of a particular edge of a hypercube being cut. In this paper we prove a wide variety of results. We present some general results on the parity of the number of edges cut. We compute probabilities of a homogeneous plane cutting one edge and several types of pairs of edges, often using geometry to obtain a simple expression for these probabilities. We then extend these results to non-homogeneous planes by developing a probability-preserving bijection between classes of homogeneous planes and classes of non-homogeneous planes a dimension lower. Using our probability formulas, we compute some important expectations related to this problem, including the number of planes required to cut a given edge with high probability and the number of planes required to cut all of the edges with high probability. Finally, we describe a sequence of points which share the same probability across dimensions, and conjecture that these are the only such points.

2 Background

We begin by discussing some of the general notation and definitions that will be used throughout this paper. Firstly, a hypercube is defined, for our purposes, as a "geometric graph on [the] vertex set $v_d = \{-1, +1\}^d \subseteq \mathbb{R}^d$ with $d2^{d-1}$ undirected edges" [4], which is centered about the origin. Next, d specifies the dimension in which a hyperplane exists and n refers to the number of hyperplanes being discussed. A cut, c, is defined as intersection of an edge and a hyperplane where the point of intersection falls distinctly within the endpoints of the edge. In other words, an intersection is not a cut if it falls on a vertex of the hypercube. We represent our homogeneous hyperplanes as d-tuples of random variables, $\mathbf{a} = (a_1, a_2, ..., a_d)$ and our non-homogeneous hyperplanes as d-tuples with one additional coordinate describing the distance from the origin, $(a_1, a_2, ..., a_d)$;t. The hyperplane is the set of points $\mathbf{x}$ given by the expression $\mathbf{a} \cdot \mathbf{x} = t$ where $t = 0$ in the homogeneous case.

In any dimension, there are infinitely many possible hyperplanes. However, in this problem, we are only concerned with which edges a hyperplane cuts. We say that two hyperplanes are equivalent if they cut the same edges. Thus, we can take the infinitely many planes and divide them into finitely many equivalence classes based on which edges they cut. Thus, although it appears continuous, we are actually studying a discrete problem. We can further characterize the equivalence classes of hyperplanes [4]. Any hyperplane will divide the vertices of the hypercube into two parts, based on which side of the hyperplane the vertices lie. We say that the hyperplane partitions the vertices of the hypercube into two sets, each of which forms a connected subgraph of the graph of the hypercube. Ziegler calls each of these subgraphs a cut-complex. The way the hyperplane partitions the vertices uniquely determines which edges are cut, as the edges that are cut are those that are exterior to the subgraphs. It is possible to write a set of linear inequalities representing which side of the hyperplane a vertex lies. Let P and Q be the sets of vertices which lie on each side of the hyperplane. Then any hyperplane with normal vector $\mathbf{a} \cdot (a_1, ..., a_d)$ and non-homogeneous coordinate t which partitions the points in this way will satisfy either

$$\mathbf{a} \cdot \mathbf{p} > t \, \forall \mathbf{p} \in P$$

and

$$\mathbf{a} \cdot \mathbf{p} < t \, \forall \mathbf{p} \in Q$$

or

$$\mathbf{a} \cdot \mathbf{p} < t \, \forall \mathbf{p} \in P$$

and

$$\mathbf{a} \cdot \mathbf{p} > t \, \forall \mathbf{p} \in Q \, .$$

Multiplying one of these by a negative number will yield the other, so the set of points in $\mathbb{R}^{d+1}$ which represents this partition of the vertices will form two regions, each of which is a reflection of the other about the origin. Moreover, since each of the two regions is described by a set of linear inequalities, each of the two regions will be a polyhedron. Since each inequality is unchanged by positive scalar multiplication, the two regions will be polyhedral cones, radiating outward. Thus, these cones divide $\mathbb{R}^{d+1}$ (excluding the boundaries of the cones) into pairs of polyhedral cones representing classes of hyperplanes. If we consider only homogeneous planes, we have exactly the geometry except in dimension $\mathbb{R}^d$. We will prove later that the homogeneous hyperplane regions and non-homogeneous hyperplane regions one dimension down will be identical. The regions correspond to picking maximal hyperplanes, i.e., hyperplanes which do not intersect the cube at a vertex, while the boundaries of the region correspond to degenerate hyperplanes, hyperplanes which cut the cube at a vertex. We follow the convention of others who work on this problem and eliminate degenerate hyperplanes from our consideration.

This geometry naturally suggests a way to pick a probability model. It is most clear for homogeneous planes, so we start with them. A homogeneous hyperplane will be determined by the direction of its normal vector. It seems logical then, to pick the normal vector uniformly in the sphere. This is equivalent to choosing each coordinate to be $N(0,1)$ and rescaling. Since the hyperplane is the same regardless of whether or not we renormalize, we omit this step unless it is helpful. For non-homogeneous planes, it seems reasonable to pick the last coordinate, like the others, to be $N(0,1)$. Thus, for our probability model, we select every random variable to be $N(0,1)$ and obtain a uniform distribution of directions (when we look at a direction as a point on the sphere).

The number of regions increases very quickly with d. However, we can reduce the complexity by taking into account the symmetries of the cube. The symmetries of the hypercube [4] are reflections in any coordinate (there are 2^d of them) and permutations of the coordinates (there are $d!$ of them) and any combinations of the above. Thus, there will be $2^d d!$ in all.

Finally, a solid angle, E, may be thought of as the projection of a surface onto the unit sphere. It is measured by the surface area of the projection on the unit sphere. For our use, we project a polygonal cone determined by three unit vectors, a_1, a_2, and a_3, in $\mathbb{R}^3$ onto the unit sphere and use the following formula for calculation [2]:

$$tan\left(\frac{E}{2}\right) = \frac{|a_1.(a_2 \times a_3)|}{1 + a_2.a_3 + a_3.a_1 + a_1.a_2} \, .$$

3 Results

3.1 Parity

In the course of our investigation we have proven several results concerning parity, which we shall describe here.

Lemma 1. *Homogeneous hyperplanes cut edges in pairs. That is, if a homogeneous hyperplane cuts one edge, it will cut the antipodal edge.*

Proof. Because a homogeneous hyperplane is not changed by multiplying its normal vector by negative one, any homogeneous hyperplane which cuts a given edge will cut the antipodal edge. □

Corollary 2. *Any number of homogeneous hyperplanes always cut an even number of edges.*

Theorem 3. *In even dimensions, any hyperplane will cut an even number of edges. In odd dimensions, the parity of the number of edges cut will be equal to the parity of the the number of edges partitioned off. In other words, hyperplanes which partition off an odd number of vertices will cut an odd number of edges while hyperplanes which partition off an even number of vertices will partition an even number of edges.*

Proof. Consider the graph of the hypercube. The vertices partitioned off by the plane will form a subgraph of the hypercube, and the edges exterior to the subgraph will be those that are cut by the plane. Since each vertex has degree d, the number of edges cut is simply $vd - 2k$, where v is the number of vertices cut off, d is the dimension, and k is the number of edges in the subgraph. From this, the above results follow. □

3.2 Probabilities of Cutting Given Edges

Here we find the probabilities of a random hyperplane cutting one or two given edges. We start by considering only homogeneous planes.

Theorem 4. *The probability of a random hyperplane cutting a given edge of the d-dimensional hypercube is given by the following integral:*

$$\frac{2}{\pi\sqrt{d-1}} \int_0^{\infty} \int_0^{y} e^{\frac{-y^2}{2}} e^{\frac{-x^2}{2d-2}} \, dxdy \, .$$

Proof. Since any edge is equivalent, we can consider only the edge $(t, -1, ..., -1)$ where $-1 < t < 1$. If a hyperplane with normal vector $\mathbf{a}$ cuts the edge, this will be equivalent to there being a solution to the following equation with t between -1 and 1:

$$a_1 t - a_2 - ... - a_d = 0$$

or

$$a_1 t = a_2 + ... + a_d \,.$$

This is equivalent to

$$-|a_1| < a_2 + ... + a_d < |a_1| \,.$$

Now, we use the change of coordinates $x = a_1$, $y = a_2 + ... + a_n$ to obtain an inequality in two normal random variables x with distribution $N(0, 1)$ and y with distribution $N(0, d-1)$:

$$-|x| < y < |x| \,.$$

Because reflecting in the first coordinate preserves the edge, the case with $x \geq 0$ yields the same probability as the case $x \leq 0$. Therefore, we need only consider $x \geq 0$ and multiply by two. Our inequality becomes

$$-x < y < x \,.$$

We wish to integrate the p.d.f. of x and y over the regions described by this equality and $x \geq 0$. Thus, the probability is:

$$\begin{aligned} 2 \int_0^\infty \int_{-y}^{y} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \frac{1}{\sqrt{2\pi(d-1)}} e^{\frac{-x^2}{2d-2}} \, dx dy \\ = \frac{1}{\pi\sqrt{d-1}} \int_0^\infty \int_{-y}^{y} e^{\frac{-y^2}{2}} e^{\frac{-x^2}{2d-2}} \, dx dy \\ = \frac{2}{\pi\sqrt{d-1}} \int_0^\infty \int_0^{y} e^{\frac{-y^2}{2}} e^{\frac{-x^2}{2d-2}} \, dx dy \,. \end{aligned}$$

□

We empirically tested this formula by generating random hyperplanes and found it to be correct up to the error in our simulation. It turns out that there is a simpler expression for the above integral

Theorem 5. *The probability of a random hyperplane cutting a given edge is*

$$\frac{2}{\pi} \tan^{-1}\left(\frac{1}{\sqrt{d-1}}\right) \,.$$

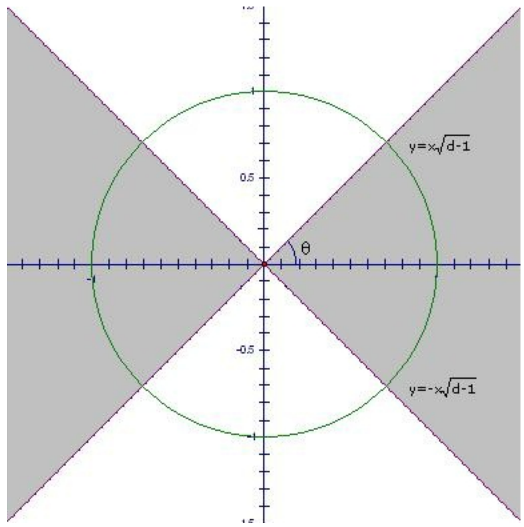


Figure 1: Regions on the Unit Circle

Proof. As we showed earlier, the region of $\mathbb{R}^d$ describing planes which hit the given edge is given by

$$-|a_1| < a_2 + ... + a_d < |a_1| \, .$$

Now, we use the change of coordinates $x = a_1$ and $y\sqrt{d-1} = a_2 + ... + a_d$ to obtain inequalities in two independent random variables with distribution $N(0,1)$:

$$-|x| < y\sqrt{d-1} < |x| \, .$$

Notice that if we wish, we can assume as above that x is larger than 0 and divide both sides by x, obtaining equations in one Cauchy random variable, which we can then easily integrate to obtain the formula. However, we prefer to use a geometric interpretation to give a more illuminating proof. The region corresponds to the region in $\mathbb{R}^2$ bounded by the lines $y = \pm x\frac{1}{\sqrt{d-1}}$ (see figure 3.2).

Thus, it remains to compute the integral of our function over that region. However, since x and y are $N(0,1)$, the integral of the p.d.f. over the conical region corresponds simply to the the proportion of the circle which lies inside the region. The length of one quarter of the length of circle inside our region is simply the angle between the line $y = x\frac{1}{\sqrt{d-1}}$ and the x-axis, or

$$\tan^{-1}\left(\frac{1}{\sqrt{d-1}}\right) \, .$$

Thus, the probability in question will be four times this length divided by the circumference of the circle:

$$4\frac{1}{2\pi}\tan^{-1}\left(\frac{1}{\sqrt{d-1}}\right) = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{d-1}}\right) \, .$$

□

Now, we do similar calculations for parallel edges.

Theorem 6. *The probability of a random plane cutting two given parallel edges with Hamming distance k is given by*

$$\frac{2}{\pi\sqrt{2\pi k(d-k-1)}}\int_0^\infty \int_0^x \int_{-x+y}^{x-y} e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2k}} e^{\frac{-z^2}{2n-2k-2}}\, dzdydx\ .$$

Proof. For this proof we proceed as before, characterizing the region of interest and integrating the p.d.f.'s over this region. Assume without loss of generality that one of the edges is $(t, -1, ..., -1)$. Also, permute the dimensions until the other has the form $(u, -1, ..., -1, 1, ..., 1)$. Now, a plane which cuts both these edges will simultaneously satisfy

$$a_1 t - a_2 - ... - a_{d-k} - a_{d-k+1} - ... - a_d = 0$$

and

$$a_1 u - a_2 - ... - a_{d-k} + a_{d-k+1} + ... + a_d = 0\ ,$$

or equivalently,

$$a_1 t = a_2 + ... + a_{d-k} + a_{d-k+1} + ... + a_d$$

and

$$a_1 u = a_2 + ... + a_{d-k} - a_{d-k+1} - ... - a_d\ .$$

Since $|t|, |u| < 1$, this will occur if and only if

$$-|a_1| < a_2 + ... + a_{d-k} + a_{d-k+1} + ... + a_d < |a_1|$$

and

$$-|a_1| < a_2 + ... + a_{d-k} - a_{d-k+1} - ... - a_d < |a_1|\ .$$

Using the change of variables $x = a_1$, $y = a_{d-k+1} + ... + a_d$, $z = a_2 + ... + a_{d-k}$, with mean 0 and variances 1, k, and $d-k-1$ respectively, we get:

$$-|x| < z + y < |x|$$

and

$$-|x| < z - y < |x|\ .$$

Reflecting in the first coordinate preserves the edge making the two cases $x \geq 0$ and $x \leq 0$ equivalent. Thus, we need only consider $x \geq 0$ and multiply by two. The p.d.f. is symmetric in y and $-y$ and the region is symmetric in y and $-y$, thus we can consider only $y \geq 0$ and multiply by two again. Our inequalities become:

$$-x < z + y < x$$

and

$$-x < z - y < x\ .$$

We rewrite them as:

$$-x - y < z < x - y$$

and

$$-x + y < z < x + y\ .$$

At this point we combine our inequalities, taking the larger of the lower bounds and the smaller of the upper bounds:

$$-x + y < z < x - y\ .$$

We need to integrate the p.d.f. of x, y, and z over the region described by this inequality and the inequalities $x > 0$, and $y > 0$. Thus, the probability is:

$$4\int_0^\infty \int_0^x \int_{-x+y}^{x-y} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \frac{1}{\sqrt{2\pi k}} e^{\frac{-y^2}{2k}} \frac{1}{\sqrt{2\pi(n-k-1)}} e^{\frac{-z^2}{2n-2k-2}}\, dzdydx$$
$$= \frac{2}{\pi\sqrt{2\pi k(d-k-1)}} \int_0^\infty \int_0^x \int_{-x+y}^{x-y} e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2k}} e^{\frac{-z^2}{2n-2k-2}}\, dzdydx\ .$$

□

We again use geometric reasoning to reduce this integral to a simpler formula.

Theorem 7. *The probability of a random homogeneous hyperplane hitting two given parallel edges with Hamming distance k is*

$$\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{k+1}\sqrt{d-k-1}+\sqrt{d-k}\sqrt{k}}\right)\ .$$

Note that this expression is symmetric in k and $d-k-1$. This is geometrically obvious, since homogeneous hyperplanes will always hit pairs of antipodal edges, and if the Hamming distance between two edges is k, then the Hamming distance between one of the edges and the antipodal of the other edge will be $d-k-1$. Note also that when $k = 0$, the above expression reduces to the probability of cutting one edge.

Proof. From the proof of the previous theorem, we have

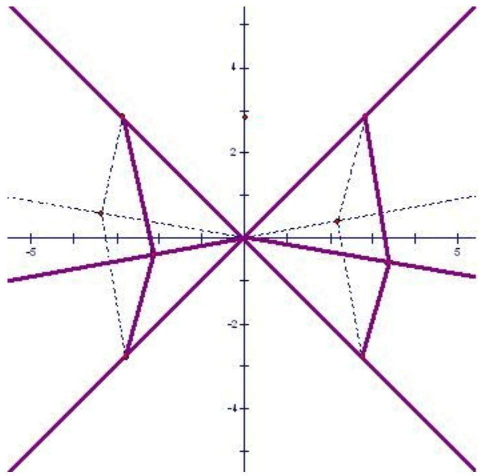
$$-|a_1| < a_2 + ... + a_{d-k} + a_{d-k+1} + ... + a_d < |a_1|$$

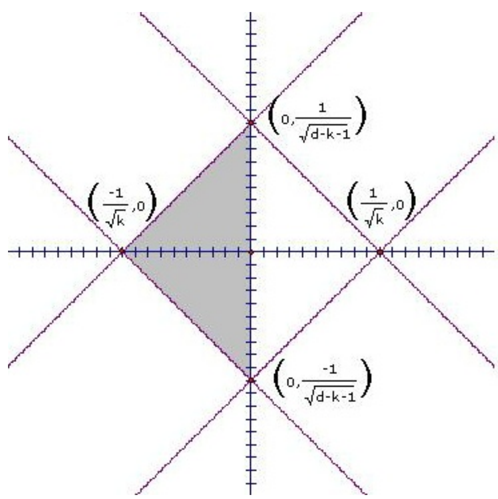
and

$$-|a_1| < a_2 + ... + a_{d-k} - a_{d-k+1} - ... - a_d < |a_1|\ .$$

We use the change of coordinates $z = a_1$, $y\sqrt{d-k-1} = a_2 + ... + a_{d-k}$, and $x\sqrt{k} = a_{d-k+1} + ... + a_d$. Notice that this results in x, y, and z being independent normal variables with $\mu = 0$ and variance $\sigma^2 = 1$. We have

$$-|z| < y\sqrt{d-k-1} + x\sqrt{k} < -|z|$$





and

$$-|z| < y\sqrt{d-k-1} - x\sqrt{k} < -|z| \, .$$

This corresponds to two polyhedral cones in $\mathbb{R}^3$ which are reflections of each other about the origin (see figure 3.2).

For simplicity we consider only the cone where z is positive and multiply by two. We depict a cross-section below when $z = 1$ (see figure 3.2).

As is clear from the figure, we need only consider one half of the diamond, then multiply by two. Integrating the p.d.f. over the described region is equivalent to computing the solid angle of the region. We use the formula for the solid angle of the region spanned by three unit vectors in terms of the dot products and the triple products of the spanning vectors:

$$2\tan^{-1}\left(\frac{2a \cdot (b \times c)}{1 + a \cdot b + b \cdot c + a \cdot b}\right) \, .$$

From our cross-section, we know that we can choose for our three vectors the following:

$$\left(0, \frac{1}{\sqrt{d-k-1}}, 1\right), \left(-\frac{1}{\sqrt{k}}, 0, 1\right), \left(0, -\frac{1}{\sqrt{d-k-1}}, 1\right) .$$

Normalizing, we have

$$\left(0, \frac{1}{\sqrt{d-k}}, \frac{\sqrt{d-k-1}}{\sqrt{d-k}}\right), \left(-\frac{1}{\sqrt{k+1}}, 0, \frac{\sqrt{k}}{\sqrt{k+1}}\right), \left(0, \frac{1}{\sqrt{d-k}}, \frac{\sqrt{d-k-1}}{\sqrt{d-k}}\right)$$

Now we plug our vectors into the the solid-angle formula to obtain (with the help of Maple)

$$2\tan^{-1}\left(\frac{1}{\sqrt{d-k-1}\sqrt{k+1}+\sqrt{d-k}\sqrt{k}}\right) .$$

Multiplying by four and dividing by the surface area of the sphere gives

$$\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{k+1}\sqrt{d-k-1}+\sqrt{d-k}\sqrt{k}}\right) .$$

□

Theorem 8. *The probability of a random hyperplane cutting two given edges which share a vertex is given by*

$$\frac{1}{\pi}\int_0^\infty \int_0^\infty \int_{-x-y}^{-|-x+y|} \frac{1}{\sqrt{2\pi(d-2)}} e^{\frac{-z^2}{2d-4}} e^{\frac{-y^2}{2}} e^{\frac{-x^2}{2}}\, dzdydx .$$

Proof. We shall begin by considering the edges $(t, -1, ..., -1)$ where $(-1 < t < 1)$ and $(-1, ..., -1, u)$ where $-1 < u < 1$. A plane which cuts both of these edges will simultaneously satisfy

$$a_1 t - a_2 - ... - a_d = 0$$

and

$$-a_1 - a_2 - ... - a_{d-1} + a_d u = 0 ,$$

or equivalently

$$a_1 t = a_2 + ... + a_d$$

and

$$a_d u = a_1 + ... + a_{d-1} .$$

Since $|t|$, $|u| < 1$, this will occur if and only if

$$-|a_1| < a_2 + ... + a_d < |a_1|$$

and

$$-|a_d| < a_1 + ... + a_{d-1} < |a_d| .$$

A change of variable with $x = a_d, y = a_1$, and $z = a_2 + ... + a_{d-1}$, with mean 0 and variances $1, 1$, and $d-2$ respectively, we get:

$$-|y| < z + x < |y|$$

and

$$-|x| < y + z < |x| \, .$$

Once again we must consider the cases $x \leq 0$ and $x \geq 0$ separately. However, since the p.d.f. is symmetric in x and $-x$, we may assume $x \geq 0$ and simply multiply our final integral by two. Doing so will change our inequalies as follows:

$$-|y| < z + x < |y|$$

and

$$-x < y + z < x \, .$$

We rearrange to obtain

$$-x - |y| < z < -x + |y|$$

and

$$-x - y < z < x - y \, .$$

We claim $y \geq 0$. If we assume $y < 0$, we obtain the following from our inequalities:

$$-x + y < z < -x - y$$

and

$$-x - y < z < x - y \, .$$

Combining these inequalities to achieve the greatest lower bound and the least upper bound, we get

$$-x - y < z < -x - y$$

which is a contradiction. Thus, $y \geq 0$ and we have:

$$-x - y < z < -x + y$$

and

$$-x - y < z < x - y \, .$$

Combining the above inequalities, we get

$$-x - y < z < -|-x + y| \, .$$

We now integrate the p.d.f. over the region described by this inequality and $x \geq 0$ and $y \geq 0$:

$$2\int_0^\infty \int_0^\infty \int_{-x-y}^{-|-x+y|} \frac{1}{\sqrt{2\pi(d-2)}} e^{\frac{-z^2}{2d-4}} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dzdydx$$
$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty \int_{-x-y}^{-|-x+y|} \frac{1}{\sqrt{2\pi(d-2)}} e^{\frac{-z^2}{2d-4}} e^{\frac{-y^2}{2}} e^{\frac{-x^2}{2}}\, dzdydx\ .$$

□

Theorem 9. *The probability that a random homogeneous hyperplane cuts two edges which share a vertex is*

$$\frac{1}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\sqrt{d-2}}{d+\sqrt{2}\sqrt{d-2}\sqrt{d-1}}\right)\ .$$

Proof. As before, we can use symmetries of the cube to transform the two edges into the form $(t, -1, -1, ..., -1)$ and $(-1, u, -1, ..., -1)$. These symmetries yield the equations

$$a_1 t = a_2 + a_3 + ... + a_d$$

and

$$a_2 t = a_1 + a_3 + ... + a_d\ ,$$

which in turn give the inequalities

$$-|a_1| < a_2 + a_3 + ... + a_d < |a_1|$$

and

$$-|a_2| < a_1 + a_3 + ... + a_d < |a_2|\ .$$

Now, we use the change of coordinates $x = a_1$, $y = a_2$, $z\sqrt{d-2} = a_3 + ... + a_d$ to obtain

$$-|x| < y + z\sqrt{d-2} < |x|$$

and

$$-|y| < x + z\sqrt{d-2} < |y|\ .$$

Subtracting y from the first line gives

$$-|x| - y < z\sqrt{d-2} < |x| - y$$

and subtracting x from the second line gives

$$-|y| - x < z\sqrt{d-2} < |y| - x\ .$$

This region divides into two parts depending on the signs of x and y. Suppose x is positive. Then taking the left side of the first set of inequalities and the right side of the second set of inequalities gives

$$-x - y < z\sqrt{d-2} < -x + |y|\ .$$

For there to be feasible z, y must be positive. Similarly, if x is negative, taking the right side of the first set of inequalities and the left side of the second set of inequalities gives

$$-|y| - x < z\sqrt{d-2} < -x - y ,$$

which means that y must be negative for there to be a feasible region. Thus, we have two polyhedral cones which are reflections of each other (since each case is closed under positive scalar multiplication and given by linear inequalities). Let us consider the cone given by $x \geq 0$, $y \geq 0$. To find the critical edges, we take all pairwise intersections of the boundary planes, i.e., the planes obtained by replacing the inequality with an equals sign. There are three planes (one is a duplicate):

$$z\sqrt{d-2} = -x - y ,$$

$$z\sqrt{d-2} = x - y ,$$

and

$$z\sqrt{d-2} = y - x .$$

Taking the intersection of the first and second planes gives the vector

$$(0, \sqrt{d-2}, -1) .$$

Taking the intersection of the first and third planes gives the vector

$$(\sqrt{d-2}, 0, -1) .$$

Taking the intersection of the second and third planes gives the vector

$$(1, 1, 0) .$$

Normalizing gives the three vectors

$$\left(0, \frac{\sqrt{d-2}}{\sqrt{d-1}}, \frac{-1}{\sqrt{d-1}}\right), \left(\frac{\sqrt{d-2}}{\sqrt{d-1}}, 0, \frac{-1}{\sqrt{d-1}}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) .$$

Now we again use the formula for the solid angle of the cone spanned by three vectors:

$$\begin{aligned} &2\tan^{-1}\left(\frac{\sqrt{2}\frac{\sqrt{d-2}}{d-1}}{1 + \frac{1}{d-1} + \sqrt{2}\frac{\sqrt{d-2}}{\sqrt{d-1}}}\right) \\ &= 2\tan^{-1}\left(\frac{\sqrt{2}\sqrt{d-2}}{d + \sqrt{2}\sqrt{d-2}\sqrt{d-1}}\right) . \end{aligned}$$

We multiply by two for the $z < 0$ case and divide by the surface area of the sphere to obtain

$$\frac{1}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\sqrt{d-2}}{d + \sqrt{2}\sqrt{d-2}\sqrt{d-1}}\right) .$$

$\square$

3.3 Non-Homogenous Planes

After exploring various probabilities of homogeneous hyperplanes cutting edges of a hypercube, it was only natural to look to non-homogenous hyperplanes. The surprising fact is that once we know the probabilities for homogeneous hyperplanes, we know the probabilities for general hyperplanes. This is because there is a probability-preserving bijection between arbitrary hyperplanes in dimension d and homogeneous hyperplanes in dimension $d+1$. In fact, the regions in R^{d+1} corresponding to each hyperplane are identical, with the appropriate understanding of the regions. Recall that a homogeneous $(d+1)$-hyperplane is determined by a $(d+1)$-tuple $(a_1, ..., a_d, a_{d+1})$ and an arbitrary d-hyperplane is determined by a d-tuple $(a_1, ..., a_d)$ and a number t which together can be viewed as a vector in $\mathbb{R}^{d+1}$.

Theorem 10. *The restriction of a homogeneous hyperplane to the $x_{d+1} = -1$ induces a bijection between equivalence classes of homogeneous $(d+1)$-planes and non-homogeneous d-planes which is given by the identity map on $\mathbb{R}^{d+1}$. In particular, there will be the same numbers of classes and probabilities of each class for both cases.*

Proof. To show this, we use a geometric argument. First, we argue that the map between equivalence classes of hyperplanes is a bijection. It is clear that the map is a bijection between hyperplanes, since the origin and an arbitrary d-hyperplane in the plane $x_{d+1} = -1$ will uniquely determine a homogeneous $(d+1)$-hyperplane. We need to show that the edges cut by a homogeneous $(d+1)$-dimensional hyperplane uniquely determine the edges cut by its restriction to the hyperplane $x_{d+1} = -1$, which is obvious, and that the edges cut by the restriction of the homogenous hyperplane to the face $x_{d+1} = -1$ uniquely determine the edges cut by the homogeneous hyperplane, which requires proof.

We use the fact that any hyperplane equivalence class is uniquely determined by the way that it partitions the vertices of the hypercube [1]. Any vertex will either lie in the face $x_{d+1} = -1$ or $x_{d+1} = 1$. The vertices in the face $x_{d+1} = -1$ will be partitioned by the given plane. Now, since edges are hit by homogeneous planes in antipodal pairs, we know exactly which edges are cut on the face $x_{d+1} = 1$ and consequently how those vertices are partitioned by the plane. Antipodal vertices will be on opposite sides of a homogeneous plane, since the line between them contains the origin, and hence a point on the plane (the regions partitioned off by a plane will be convex). Using this, we know exactly which vertices lie on the same side of the homogeneous hyperplane, and hence, which edges are cut by the hyperplane. Thus, the edges cut on one face of a hyperplane uniquely determines the edges cut by the homogeneous plane on the whole hyperplane. This gives us our bijection between classes.

Now, we show that the restriction map, induces the identity map on the coordinates of the hyperplanes in $\mathbb{R}^{d+1}$. To see this, let $(x_1, ..., x_d, -1)$ be a point on the hyperplane $x_{d+1} = -1$. Then $(x_1, ..., x_d, -1)$ will lie on the homogeneous $(d+1)$-hyperplane $(a_1, ..., a_d, t)$ if and only if

$$\begin{aligned} &\iff \quad a_1x_1 + ... + a_dx_d - t = 0 \\ &\iff \quad a_1x_1 + ... + a_dx_d = t \;, \end{aligned}$$

i.e., if and only if $(x_1, ..., x_d)$ lies on the hyperplane $(a_1, ..., a_d);t$. Thus, the restriction map induces the identity on the coordinates of the planes in $\mathbb{R}^{d+1}$.

□

The following expression allows for the direct computation of the probability of cutting a given edge in dimension d:

$$P(1\ edge\ general) = \frac{2}{\pi} tan^{-1}\left(\frac{1}{\sqrt{d}}\right).$$

This relation is interesting because it seems intuitive that the best-chosen planes would be homogenous planes, which would lessen the complexity of the edge-cutting problem. However, it is not at all clear how to prove this fact.

3.4 Expectations

Using some of our probability formulas, we were able to generate some statistics related to hyperplanes and edges. In this section, we frequently use the estimate $\tan^{-1}(x) \approx x$.

Proposition 11. *The expected number of edges cut by one hyperplanes in d dimensions is*

$$\frac{d2^d}{\pi} tan^{-1}\left(\frac{1}{\sqrt{d}}\right) \approx \frac{2^d\sqrt{d}}{\pi}\ .$$

The proof is a direct result of the additivity of expectation. We continue with another obvious result.

Proposition 12. *The probability that a given edge is cut by n hyperplanes is $1-(1-p)^n$ where*

$$p = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{d}}\right)\ .$$

We use the above definition of p freely throughout the rest of this section. Note that $p \to 0$ as $d \to \infty$.

Theorem 13. *The number of planes required to cut a given edge with high probability is $\Omega(\sqrt{d})$.*

Proof. Let n be the number of planes required, where n is a function of d. Then n planes cut the given edge with high probability if and only if

$$\begin{aligned} & 1-(1-p)^n \to 1 \\ \iff\quad & (1-p)^n \to 0\ . \end{aligned}$$

Now, reformulating the right hand side gives

$$\begin{aligned} & (1-p)^n \to 0 \\ \iff\quad & e^{n\log(1-p)} \to 0 \\ \iff\quad & n\log(1-p) \to -\infty\ . \end{aligned}$$

Now, we use the power series for $\log(1-p)$ to obtain

$$
\begin{aligned}
& n\log(1-p) \to \infty \\
\iff \quad & n\left(-p - \frac{p^2}{2} - \frac{p^3}{3} - \ldots\right) \to -\infty \\
\iff \quad & -np \to -\infty \\
\iff \quad & \frac{2n}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{d}}\right) \to \infty \\
\iff \quad & \frac{n}{\sqrt{d}} \to \infty \, .
\end{aligned}
$$

Thus, n hyperplanes will cut a given edge with high probability if and only if n grows faster than $\sqrt{d}$. □

Theorem 14. *The number of planes required to cut all edges with high probability is the smallest integer greater than*

$$\frac{d^{3/2}\pi \log 2}{2} \, .$$

Proof. The expected number of edges not cut by n hyperplanes is

$$d2^{d-1}(1-p)^n \, .$$

The probability that n planes will cut all of the edges will go to one if and only if this expression goes to zero. We proceed as before, taking logarithms:

$$
\begin{aligned}
& d2^{d-1}(1-p)^n \to 0 \\
\iff \quad & e^{\log d + (d-1)\log 2 + n\log(1-p)} \to 0 \\
\iff \quad & \log d - \log 2 + d\log 2 + n\log(1-p) \to -\infty \, .
\end{aligned}
$$

We can ignore the $\log d$ term and the $-\log 2$ term, as they grow much more slowly than d. We use the power series expansion of log, obtaining

$$d\log 2 + n(-p - \frac{p^2}{2} - \ldots) \to -\infty$$

Now, the logarithm will be dominated by the first term, but will also be less than the first term for $p > 0$. Thus, the expression goes to negative infinity if and only if the first term in the logarithm cancels out the $d\log 2$ term, i.e. if and only if

$$d\log 2 = np = n\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{d}}\right) \approx \frac{2n}{\pi\sqrt{d}}$$

or

$$n = \frac{d^{3/2}\pi\log 2}{2}$$

□

3.5 A Recurrence Relation

As we were looking at tables of probabilities of one homogeneous plane, we noticed several entries were the same across dimensions. We investigated further and were able to find and prove that there is an infinite sequence of probabilities which agree across dimensions. We hope future investigation might provide deeper insight into why such probabilities correspond, possibly leading to the classification of different types of regions. The recurrence we found is summarized in the following theorem.

Theorem 15. *Let k be fixed. Define the sequence a_{n+1} by $a_1 = 2k+1$, $a_2 = a_1^2 + k$ and $a_n = \frac{a_n^2}{a_{n-1}-k} + k$. Then $\{a_n\}$ is a sequence of relationships between probabilities. In particular, the probability of cutting two given parallel edges with a homogeneous plane in dimension a_n is the same as cutting one given edge with a homogeneous plane in dimension $a_{n+1} - k$.*

We prove this by first proving a closed formula for $\{a_n\}$.

Theorem 16. *a_n has the generating function*

$$A(x) = \frac{(k+1)x^2 + (4k^2+5k+2)x + 2k+1}{x^3 - (4k+3)x^2 + (4k+3)x - 1}\,.$$

This leads to the closed formula

$$a_n = \frac{2k+1}{2} + \frac{1}{4}t^n + \frac{1}{4}t^{-n}$$

where

$$t = 2k+1+2\sqrt{k^2+k}\,.$$

The generating function gives the additional recurrence relation

$$a_{n+1} = (4k+3)a_n - (4k+3)a_{n-1} + a_{n-2}\,.$$

Proof. First note

$$\begin{aligned}\frac{1}{t} &= \frac{1}{2k+1+2\sqrt{k^2+k}}\,\frac{2k+1-2\sqrt{k^2+k}}{2k+1-2\sqrt{k^2+k}} \\ &= \frac{2k+1-2\sqrt{k^2+k}}{4k^2+4k+1-4(k^2+k)} \\ &= 2k+1-2\sqrt{k^2+k}\,,\end{aligned}$$

which is the conjugate of t. Now, start with the generating function $A(x)$. Clearly 1 is a root of the denominator. Dividing the denominator by $x-1$ gives $x^2-(4k+2)x+1$ which, by the quadratic formula, has roots $2k+1+2\sqrt{k^2+k} = t$ and $2k+1-2\sqrt{k^2+k} = t^{-1}$. We do a partial fraction decomposition of $A(x)$:

$$A(x) = \frac{2k+1}{2}\frac{1}{1-x} + \frac{1}{4}\frac{1}{(2k+1+2\sqrt{k^2+k})-x} + \frac{1}{4}\frac{1}{(2k+1-2\sqrt{k^2+k})-x}\,.$$

We use the formula for a geometric series to obtain

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{2k+1}{2} x^n + \sum_{n=0}^{\infty} \frac{1}{4} t^n x^n + \sum_{n=0}^{\infty} \frac{1}{4} t^{-n} x^n$$
$$= \sum_{n=0}^{\infty} (\frac{2k+1}{2} + \frac{1}{4} t^n + \frac{1}{4} t^{-n}) x^n \ ,$$

which gives

$$a_n = \frac{2k+1}{2} + \frac{1}{4} t^n + \frac{1}{4} t^{-n} \ .$$

Next, we prove that this function satisfies the inter-dimensional relation described above. The probability of cutting one edge in dimension $a_{n+1} - k$ is

$$\frac{2}{\pi} \tan^{-1} \left(\frac{1}{\sqrt{a_{n+1} - k - 1}} \right) \ ,$$

while the probability of cutting two parallel edges at distance k from each other is

$$\frac{2}{\pi} \tan^{-1} \left(\frac{1}{\sqrt{k+1}\sqrt{a_n - k - 1} + \sqrt{a_n - k}\sqrt{k}} \right) \ .$$

These will be equal if and only if

$$\sqrt{a_{n+1} - k - 1} = \sqrt{k+1}\sqrt{a_n - k - 1} + \sqrt{a_n - k}\sqrt{k} \ .$$

We compute the square of each side (since each side is positive, this does not introduce any new solutions). The square of the left-hand side is

$$a_{n+1} - k - 1 = -\frac{1}{2} + \frac{1}{4} t^{n+1} + \frac{1}{4} t^{-(n+1)} \ .$$

The square of the right hand side is

$$\begin{aligned}
&\left(\frac{1}{2} + \frac{1}{4}(t^n + t^{-n})\right) k + \left(-\frac{1}{2} + \frac{1}{4}(t^n + t^{-n})\right)(k+1) + \\
&2\sqrt{k^2+k}\sqrt{\left(\frac{1}{2} + \frac{1}{4}(t^n + t^{-n})\right)\left(-\frac{1}{2} + \frac{1}{4}(t^n + t^{-n})\right)} \\
= \quad & -\frac{1}{2} + (2k+1)\left(\frac{1}{4} t^n + \frac{1}{4} t^{-n}\right) + \frac{2\sqrt{k^2+k}}{4}\sqrt{t^{2n} - 2 + t^{-2n}} \\
= \quad & -\frac{1}{2} + (2k+1)\left(\frac{1}{4} t^n + \frac{1}{4} t^{-n}\right) + \frac{2\sqrt{k^2+k}}{4}(t^n - t^{-n}) \\
= \quad & -\frac{1}{2} + \frac{1}{4}(2k+1+2\sqrt{k^2+k}) t^n + \frac{1}{4}(2k+1-2\sqrt{k^2+k}) t^{-n} \\
= \quad & -\frac{1}{2} + \frac{1}{4} t^{n+1} + \frac{1}{4} t^{-(n+1)} \ .
\end{aligned}$$

Thus, the sequence given by the generating function satisfies the inter-dimensional relation.

We now prove that they agree with our original sequence, by simply showing that they satisfy our recurrence relation. Note first that a_1 as given by the formula is $2k+1$. A little more computation shows that a_2 also agrees with the above initial conditions. Now, in order to prove the recurrence relation, we need to show

$$a_{n+1} = \frac{a_n^2}{a_{n-1} - k} + k \; ,$$

or equivalently

$$(a_{n+1} - k)(a_{n-1} - k) = a_n^2 \; .$$

The left hand side is

$$\begin{aligned}
&= \frac{1}{16}(4 + 2t^{n+1} + 2t^{-(n+1)} + 2t^{n-1} + t^{2n} + t^{-2} + 2t^{-(n-1)} + t^2 t^{-2n}) \\
&= \frac{1}{16}(4 + t^2 + t^{-2} + 2t(t^n + t^{-n}) + 2t^{-1}(t^n + t^{-n}) + t^{2n} + t^{-2n}) \\
&= \frac{1}{16}(16k^2 + 16k + 6 + t^{2n} + t^{-2n} + 2(t + t^{-1})(t^n + t^{-n})) \\
&= \frac{1}{16}(16k^2 + 16k + 6 + t^{2n} + t^{-2n} + 4(2k+1)(t^n + t^{-n})) \; .
\end{aligned}$$

The right hand side is

$$\begin{aligned}
&= \frac{1}{16}(16k^2 + 16k + 4 + t^{2n} + t^{-2n} + 4(2k+1)(t^n + t^{-n}) + 2) \\
&= \frac{1}{16}(16k^2 + 16k + 6 + t^{2n} + t^{-2n} + 4(2k+1)(t^n + t^{-n})) \; .
\end{aligned}$$

□

4 Conclusion

In the future, we hope to be able to compute the minimum number of hypercubes to cut all of the edges on the 7-cube and possibly the 8-cube. In fact, we are currently working compute the 7-cube and hope to have these results in the near future. Ideally, we would like to find a generalization for the cut numbers.

Regarding our work with the probability of cutting two given edges, we hope to complete our tables by finding an efficient method for generating the probabilities of cutting two skew edges. Currently we are only able to estimate these values and we are only able to do so for low dimensions.

5 Acknowledgements

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