COMPUTATION OF MODULAR FORMS OF WEIGHT 3/2

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Abstract. In this paper we describe a computational investigation (via SAGE) into the basis problem for modular forms of weight 3/2. Serre and Stark resolve the basis problem at weight 1/2 in terms of lifts and twists of the classical theta function [SS77]. By analogy, we consider whether a basis can be constructed at weight 3/2 from lifts and twists of forms constructed from ternary quadratic forms. We find that this does not generate a basis, and so we investigate the effect of adding in forms created from multiplying weight 1/2 forms against weight 1 forms. This too does not generate a basis. Let the subspace of $M_{3/2}(N,\chi)$ generated in this way be denoted $A_{3/2}(N,\chi)$. We consider the effect of applying Hecke operators; this too is not sufficient to generate a basis, and additionally does not generate any new linearly-independent forms. We offer a series of conjectures on the linear-dependence of certain classes of forms. We also include an appendix of the dimension of all subspaces $A_{3/2}(N,\chi)$ for $N \leq 600$ and χ a quadratic character.

Half-Integral-Weight Modular Forms, Basis Problem, Computation, SAGE

1. Introduction

It is known that the set of modular forms of a given weight, level, and character is a finite dimensional complex vector space. The "basis problem" for modular forms is the problem of finding a basis for these spaces with elements whose Fourier coefficients may be explicitly computed. In [Shi73], Shimura asks whether it is possible to find a basis for the weight 1/2 case using theta series. Serre and Stark answer yes and provide a proof in [SS77]. For integral weight spaces, the basis problem has been solved for all spaces of modular forms (see [Miy89, pp. 176-179] and [HPS89]). It has been solved for the space of cusp forms as well [HPS89]. This paper explores the basis problem for spaces of weight 3/2 forms from a computational perspective. In particular, it describes the various tools and methods used to compute as much of a basis as possible for various weight 3/2 spaces.

2. Background

2.1. Definitions.

2.1.1. *The Action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$.* The following notation is standard. Let $\mathbb{H}$ denote the upper half plane, viz., $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$; and let Γ the familiar group $SL_2(\mathbb{Z})$. It is not difficult to see that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\gamma z := \frac{az+b}{cz+d}$$

defines a group action on $\mathbb{H}$, i.e.,

(1) $z \in \mathbb{H}, \gamma \in \Gamma \Rightarrow \gamma z \in \mathbb{H}$,
(2) $Iz = z$,
(3) $(\gamma_1\gamma_2) z = \gamma_1(\gamma_2 z) \quad \forall \gamma_1, \gamma_2 \in \Gamma, \forall z \in \mathbb{H}$.

Of course, as is always the case with a group action, the action of Γ induces an equivalence relation on $\mathbb{H}$. Thus, one would like to find a representative for each equivalence class. That is, we would like a fundamental domain for Γ. A fundamental domain for Γ is a closed subset $F \subset \mathbb{H}$ such that

(1) for all $z \in \mathbb{H}$, z is Γ-equivalent to a point in F; but
(2) if $z_1, z_2 \in F$ are Γ-equivalent, then $z_1 = z_2$ or z_1, z_2 are on the boundary of F.

Proposition 1. *Let* $F := \left\{z \in \mathbb{H} : -\frac{1}{2} \leq \mathrm{Re}\,(z) \leq \frac{1}{2}, |z| \geq 1\right\}$. *Then* F *(see Figure 1) is a fundamental domain for* Γ.

Proof. See [Kob93, Prop. 3.1.1]. □

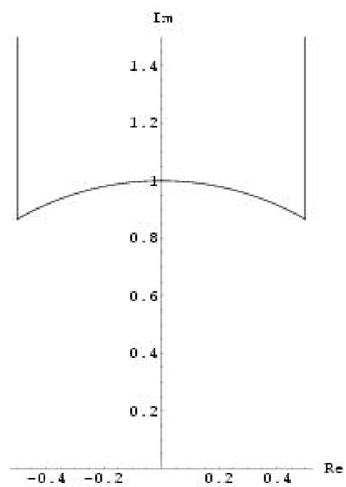

FIGURE 1. F

2.1.2. *The Compactification of* $\mathbb{H}$ *and Congruence Subgroups of* Γ. Since it will be more convenient to work in a compact space, we also adjoin a point "at infinity", which we think of as being far up the imaginary axis. Observe that

$$\gamma\infty := \lim_{z\to\infty} \frac{az+b}{cz+d} = a/c.$$

Furthermore, since $\gamma \in \Gamma$, it follows that $\det \gamma = 1$. So given any $a/c \in \mathbb{Q}$ in lowest terms (i.e. $(a, c) = 1$), we can easily find b and d so that $ad - bc = 1$. Thus, we can construct a matrix that sends ∞ to a/c. In this way, we can see that every rational number is Γ-equivalent to ∞. Thus, we take $\overline{\mathbb{H}} := \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ to be the compactification of $\mathbb{H}$; and we let $\overline{F} := \overline{\mathbb{H}}/\Gamma$. The equivalence classes of $\mathbb{Q} \cup \{\infty\}$ are referred to as *cusps*. Since all rational numbers and ∞ are Γ-equivalent, we have just one cusp for $\mathbb{H}/\Gamma$, and we take ∞ as the class representative. However, it is easy enough to see that for subgroups of Γ, there may be more than one equivalence class among the elements of $\mathbb{Q} \cup \{\infty\}$. In such a case, we select a representative from each class.

Let $N \in \mathbb{N}$. A very important subgroup of Γ is the following:

$$\Gamma_0\,(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

We say that $\Gamma_0\,(N)$ is a congruence subgroup of level N, and it is easy to see that $\Gamma_0\,(N) \leq \Gamma$. It is also clear that for any $N \in \mathbb{N}$, $[\Gamma : \Gamma_0\,(N)] < \infty$ (see [Kil08, pp.23-24]), and thus $\overline{\mathbb{H}}/\Gamma_0\,(N)$ will have at most $[\Gamma : \Gamma_0(N)]$ equivalence classes among the cusps. However, it turns out that this is a rather poor bound on the number of cusps.

Of course, it would be nice to find a fundamental domain for this special subgroup as well. The following proposition helps us find fundamental domains for subgroups of finite index in terms of F, the fundamental domain for Γ. For a proof, see [Kob93, pp. 105-107].

Proposition 2. *Let $\overline{F}$ be the fundamental domain for the action of Γ on $\overline{\mathbb{H}}$ and let $\Gamma_0(N) \leq \Gamma$ have finite index $[\Gamma : \Gamma_0(N)] = n$. Suppose $\{\alpha_j\}_{j=1}^{n}$ is a complete set of coset representatives for $\Gamma_0(N)$ in Γ. Then $F_0(N) := \bigcup_{j=1}^{n} \alpha_j^{-1}\overline{F}$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\overline{\mathbb{H}}$.*

As an example, we compute $F_0(4)$, the fundamental domain for $\Gamma_0(4)$. Define T and S as follows:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It can be checked that a complete set of coset representatives for $\Gamma_0(4)$ is

$$\left\{I, S, T^{-1}S, T^{-2}S, T^{-3}S, ST^{-2}S\right\}.$$

Koblitz leaves this as an exercise on p. 107 in [Kob93]. Thus, we obtain the fundamental domain for $\Gamma_0(4)$ pictured in Figure 2.

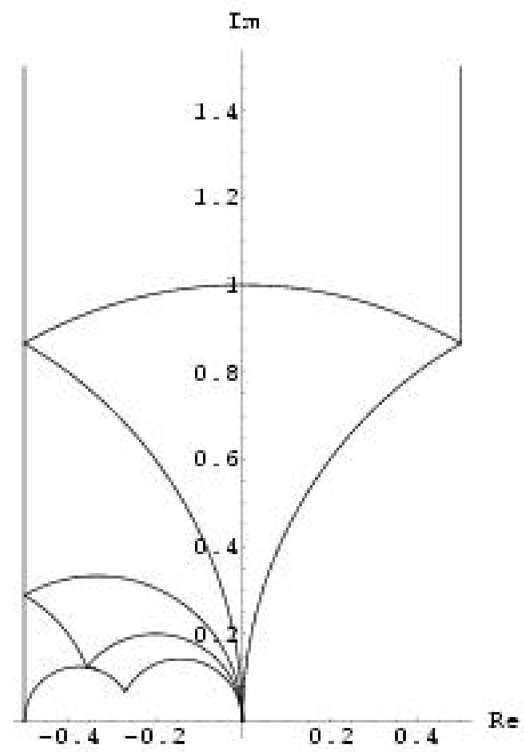

FIGURE 2. $\overline{F_0(4)}$

The appearance of Figure 2 suggests that $\overline{F_0(4)}$ contains 3 cusps: $\{\infty, -1/2, 0\}$. This is in fact true. However, since these points fall on the boundary, one must check to be certain that they are not $\Gamma_0(4)$-equivalent. This is not difficult.

2.1.3. *Periodic Functions on* $\overline{\mathbb{H}}$ *and Holomorphicity at Infinity.* Let $N \in \mathbb{N}$. Then we make the definition $q_N := e^{2\pi iz/N}$ (and when $N = 1$, we simply put $q := q_1$). If f is a function on $\mathbb{H}$ of period N, then from Fourier analysis we know that f can be expressed as

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q_N^n.$$

Now note that $\lim_{z \to i\infty} q_N = \lim_{z \to i\infty} e^{2\pi iz/N} = 0$. In light of this, we say that the series $\sum_{n=0}^{\infty} a_n q_N^n$ is the expansion of f about ∞. Furthermore, we say that f is *holomorphic at infinity* if $a_n = 0$ for $n < 0$, and f is said to *vanish at infinity* if $a_n = 0$ for $n \leq 0$.

2.1.4. *Integral and Half-Integral Weight Modular Forms for* $\Gamma_0(N)$. We are interested in functions which satisfy certain transformation laws under the action of $\Gamma_0(N)$ on $\mathbb{H}$. Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic, and let $k \in \frac{1}{2}\mathbb{Z}$. In the case that k is not integral, we put $z^k := (\sqrt{z})^{2k}$, where $\sqrt{\cdot}$ denotes the principal branch of the square root. We let $\left(\frac{c}{d}\right)$ denote Kronecker's generalization of the Legendre symbol except that we put $\left(\frac{0}{\pm 1}\right) := 1$. Now, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We define the *weight* k *operator* $[\gamma]_k$ by

$$(f[\gamma]_k)(z) = \begin{cases} (cz+d)^{-k} f(\gamma z), & k \in \mathbb{Z} \\ \left(\frac{c}{d}\right)^{-2k} \left(\frac{-1}{d}\right)^{k} (cz+d)^{-k} f(\gamma z), & k \in \frac{1}{2} + \mathbb{Z} \end{cases}$$

Note that the definition differs depending on whether k is integral or half-integral.

Definition 1. Let $k \in \frac{1}{2}\mathbb{Z}$, and let χ be a Dirichlet character modulo N, where $4 \mid N$ if k is not integral. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be a *modular form* of weight k and character χ for $\Gamma_0(N)$ if

(1) $f[\gamma]_k = \chi(d) f$ for all $\gamma \in \Gamma_0(N)$;
(2) $f[\xi]_k$ possesses a Fourier expansion of the form

$$(f[\xi]_k)(z) = \sum_{n=1}^{\infty} a_{\xi,n} q_N^n$$

for all $\xi \in \Gamma$, where $q_N := e^{\frac{2\pi iz}{N}}$.

If, in addition, $a_{\xi,0} = 0$ for all $\xi \in \Gamma$, we say that f is a *cusp form*.

Condition (1) tells us that modular forms satisfy a large family of transformation formulas. Condition (2) is referred to as holomorphicity (resp. vanishing) at the cusps of $\mathbb{H}/\Gamma_0(N)$. This is because if $x \in \mathbb{Q}$ is not $\Gamma_0(N)$-equivalent to infinity, we choose a coset representative γ for $\Gamma/\Gamma_0(N)$ such that $\gamma\infty = x$. It can be shown that condition (2) does not depend on the choice of coset representative. For proof of this fact, see [Kob93, Prop. 3.3.16].

Now, let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and note that for every $N \in \mathbb{N}$, we have that $T \in \Gamma_0(N)$. Thus any modular form for $\Gamma_0 a(N)$ will satisfy $f(z) = f(z+1)$. This means these modular forms will have $q-$expansions at ∞, instead of just q_N-expansions.

For $k \in \frac{1}{2}\mathbb{Z}$, we denote the class of all modular forms (resp. cusp-forms) of weight k and character χ for $\Gamma_0(N)$ by $M_k(N, \chi)$ (resp. $S_k(N, \chi)$). Also, whenever χ is the trivial character, we usually just write $M_k(N)$ and $S_k(N)$. It is well known that these sets actually form finite dimensional vector spaces over $\mathbb{C}$. See [Kob93] for a more detailed exposition of this fact.

Definition 2. Let χ be a Dirichlet character. Then χ is said to be *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

Notice that every character is either odd or even since $(-1)^2 = 1$ implies that -1 must be sent to a second root of unity by χ. In fact, we have the following result.

Fact 1. If k is an integer, then $M_k(N,\chi)$ contains a nontrivial element only if k and χ are both even or both odd.

Proof. See [Ono04, Rem. 1.17 pp. 4] □

For the case of spaces of half-integral weight modular forms, we have a similar result.

Fact 2. Suppose $M_{k/2}(N,\chi)$ contains a nontrivial element. Then χ is an even character.

Proof. We can take $\gamma = -I \in \Gamma_0(N)$. Then we can immediately see $f(z) = f(-Iz) = \chi(-1) f(z)$, which implies $\chi(-1) = 1$. □

2.1.5. *The Space of Weight 1/2 Modular and Cusp Forms for* $\Gamma_0(N)$. In [SS77], Serre and Stark prove that a basis for the spaces $M_{1/2}(N,\chi)$ and $S_{1/2}(N,\chi)$ can be constructed quite simply. Let

$$\theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

be the classical theta function. Let ψ be an even ($\psi(-1) = 1$) primitive Dirichlet character of conductor r, and let $t \in \mathbb{N}$. Now define

$$\theta_{\psi,t}(z) := \sum_{n\in\mathbb{Z}} \psi(n) q^{tn^2}.$$

Then it can be shown that $\theta_{\psi,t} \in M_{1/2}\left(4r^2 t, \left(\frac{t}{\cdot}\right)\psi\right)$.

We let $\Omega(N,\chi)$ denote the pairs (ψ,t) such that

(1) $4r^2 t \mid N$, and
(2) $\chi(n) = \psi(n)\left(\frac{t}{n}\right)$ for every n coprime to N.

Theorem 1. [Ono04, pp. 12-13] *The set of theta series of the form* $\theta_{\psi,t} = \sum_{n\in\mathbb{Z}} \psi(n) q^{tn^2}$*, with* $(\psi,t) \in \Omega(N,\chi)$*, make up a basis for* $M_{1/2}(N,\chi)$*.*

Corollary 1. [Ono04, pp. 12-13] *The set of theta series of the form* $\theta_{\psi,t} = \sum_{n\in\mathbb{Z}} \psi(n) q^{tn^2}$*, as* (ψ,t) *varies over the elements* $\Omega(N,\chi)$ *for which* ψ *is not totally even, forms a basis for* $S_{1/2}(N,\chi)$*.*

From Theorem 4 and Corollary 2, we can create an algorithm for calculating the dimensions of $M_{1/2}(N,\chi)$ and $S_{1/2}(N,\chi)$.

2.2. **Dimension Formulas.** For integer and half-integer weight spaces, the dimension formulas are different. In the integer weight case, the formula is as follows.

Theorem 2. [CO77, Thm 1] *If* $k \in \mathbb{Z}$ *and* χ *is a Dirichlet character modulo* N *for which* $\chi(-1) = (-1)^k$*, then*

$$\begin{aligned}\dim(S_k(N,\chi)) - \dim(M_{2-k}(N,\chi)) = \frac{(k-1)N}{12}\prod_{p\mid N}\left(1+p^{-1}\right) - \frac{1}{2}\prod_{p\mid N}\lambda(r_p,s_p,p) \\ + \nu_k \sum_{\substack{x \pmod N \\ x^2+1\equiv 0 \pmod N}} \chi(x) + \mu_k \sum_{\substack{x \pmod N \\ x^2+x+1\equiv 0 \pmod N}} \chi(x),\end{aligned} \tag{1}$$

where

$$\lambda\left(r_p, s_p, p\right) = \begin{cases} p^{r'} - p^{r'-1} & \text{if } 2s_p \leq r_p = 2r' \\ 2p^{r'} & \text{if } 2s_p \leq r_p = 2r'+1 \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p \end{cases}$$

$$\nu_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ -1/4 & \text{if } k \equiv 2 \pmod 4 \\ 1/4 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

$$\mu_k = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3 \\ -1/3 & \text{if } k \equiv 2 \pmod 3 \\ 1/3 & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

For half-integer weight spaces, the dimension formula is slightly different. The formula may be found in [CO77, Thm 2] or [Ono04, pp. 16].

Theorem 3. *Let $k \in \frac{1}{2} + \mathbb{Z}$ (so $4 \mid N$ and $\chi(-1) = 1$). We have*

$$\dim S_k\left(\Gamma_0\left(N\right), \chi\right) - \dim M_{2-k}\left(\Gamma_0\left(N\right), \chi\right) = \frac{k-1}{12} N \prod_{p \mid N}\left(1 + \frac{1}{p}\right) - \frac{\zeta\left(k, N, \chi\right)}{2} \prod_{p \mid N, p \neq 2} \lambda\left(r_p, s_p, p\right),$$

where r_p indicates the exponent of p in the prime factors of N and s_p indicates the exponent of p in the prime factors of the conductor of χ, with λ as defined above and ζ defined as follows:

If $r_2 \geq 3$, then

$$\zeta\left(k, N, \chi\right) = \begin{cases} \lambda\left(r_2, s_2, 2\right) & \text{if } r_2 \geq 4 \\ 3 & \text{if } r_2 = 3 \end{cases}$$

If $r_2 = 2$ and there exists a prime p with $p \mid N$ and $p \equiv 3 \pmod 4$ such that either r_p is odd or $0 < r_p < 2s_p$, then let

$$\zeta\left(k, N, \chi\right) = 2.$$

Else, we have $r_2 = 2$ and every prime $p \equiv 3 \pmod 4$ with $p \mid N$ (if any exist) has the property that r_p is even and $r_p < 2s_p$. In these cases let

$$\zeta\left(k, N, \chi\right) = \begin{cases} 3/2 & \text{if } k - \frac{1}{2} \in 2\mathbb{Z} \text{ and } s_2 = 0 \\ 5/2 & \text{if } k - \frac{1}{2} \in 2\mathbb{Z} \text{ and } s_2 = 2 \\ 5/2 & \text{if } k - \frac{3}{2} \in 2\mathbb{Z} \text{ and } s_2 = 0 \\ 3/2 & \text{if } k - \frac{3}{2} \in 2\mathbb{Z} \text{ and } s_2 = 2 \end{cases}$$

Notice how both of these formulas relate the dimension of the space of weight k to the related space of weight $2-k$. Then, to calculate any dimensions using the formulas, we need a few more ideas. The following fact is mentioned in [CO77, pp. 71]

Fact 3. [CO77] For $k \in \frac{1}{2}\mathbb{Z}$, $\dim M_k\left(N, \chi\right) = \dim S_k\left(N, \chi\right) = 0$ for $k \leq 0$, except for the case $\dim M_0\left(N, \chi_0\right) = 1$.

Our project focuses on spaces of forms of weight $3/2$. Notice that the dimension formulas relate the spaces of weight $3/2$ to spaces of weight $1/2$, so before finding the dimensions for weight $k = 3/2$, we need to first find the dimensions for weight $k = 1/2$. Recall that from [SS77], we have a way of producing a basis for $M_{1/2}\left(N, \chi\right)$ and $S_{1/2}\left(N, \chi\right)$. We can therefore create an algorithm for calculating the dimensions of these spaces which iterates through the possible

$(\psi, t) \in \Omega(N, \chi)$ (where $\Omega(N, \chi)$ is as defined on Page 5) and counts the pairs that have all the necessary qualifications. Now we can easily create an algorithm to calculate the dimension for spaces of forms of weight 3/2. In fact, the dimension formulas in conjunction with Fact 3 allow us to calculate the dimension for every space of modular and cusp forms of a given weight $k \neq 1$, level N, and character χ.

2.3. **Quadratic Forms and Theta Series.** A quadratic form Q in k variables is a degree two polynomial of the form:

$$Q(x) = Q(x_1, x_2, \ldots, x_k) = \sum_{i,j} c_{i,j} x_i x_j = x^T C x,$$

where

$$C = [c_{i,j}] \text{ and } x = [x_1, x_2, \ldots, x_k]^T.$$

If $Q(x) > 0$ for all $x \in \mathbb{R}^k \backslash \left\{\vec{0}\right\}$, then Q is called *positive definite*. All quadratic forms in this paper will be assumed to be positive definite.

In [Leh92], we find the following result, which is a special case of Shimura's theorem found in [Shi73].

Theorem. [Shi73] *Let $Q(x_1, \ldots, x_k)$ be a positive definite quadratic form having integer coefficients. Let A be the $k \times k$ matrix*

$$A_Q = \left[\frac{\partial^2 Q}{\partial x_i \partial x_j}\right].$$

Define N to be the smallest positive integer so that NA^{-1} is an even matrix (i.e. a matrix with integral entries, and even integers on the main diagonal). Let

$$\theta_Q(z) := \sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^k} q^{Q(m_1, \ldots, m_k)},$$

where $q = e^{2\pi i z}$.

Then $\theta_Q(z) \in M_{k/2}(N, \chi_\Delta)$, where $\Delta = \begin{cases} \det(A), & \text{if } k \equiv 0 \pmod 4, \\ -\det(A), & \text{if } k \equiv 2 \pmod 4, \\ \det(A)/2, & \text{if } k \text{ is odd.} \end{cases}$

As a trivial example of Shimura's theorem, we consider the case where $k = 1$. There is only one positive definite primitive quadratic form in one variable, namely $Q(x) = x^2$. Then $A_Q = [2]$, so that $A_Q^{-1} = [1/2]$. This establishes that $N = 4$. Also, since k is odd, χ_Δ is trivial. Consider the series θ_Q. This is easy to compute:

$$\theta_Q(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + \sum_{n=1}^{\infty} 2q^{n^2}$$

We refer to $\theta_Q(z)$ in this case simply as $\theta(z)$, the classical theta function. By Shimura's theorem, $\theta(z) \in M_{1/2}(4)$.

Clearly, finding the n-th Fourier coefficient of θ_Q amounts to finding the number of distinct ways that the quadratic form Q can represent the integer n. Two quadratic forms are said to be *equivalent* if they represent the same integers in the same number of ways. More precisely, if f, g are k-ary quadratic forms, then f is equivalent to g if there exists a linear change of variables between them. That is, $f \sim g$ if there exists $A \in SL_2(\mathbb{Z})$ such that $f(\vec{x}) = g(\vec{x}A)$. Since equivalent quadratic forms will determine the exact same theta series, it makes since to look for a canonical representative for each equivalence class. Such a representative will be called *reduced*.

For our purposes, binary and ternary quadratic forms will be most important. We now give the definition for reduced binary and ternary forms.

Definition 3. A binary quadratic form $f(x) = ax^2 + bxy + cy^2$ is said to be **reduced** if

(1) $|b| \leq a \leq c$, and
(2) $b \geq 0$ whenever $a = |b|$, or $a = c$.

A ternary quadratic form $g(x) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ is said to be **reduced** if

(1) $a \leq b \leq c$,
(2) r, s, t are all positive or all nonpositive,
(3) $a \geq |t|;\ a \geq |s|;\ b \geq |r|$,
(4) $a + b + r + s + t \geq 0$,
(5) $a = t \Rightarrow s \leq 2r;\ a = s \Rightarrow t \leq 2r;\ b = r \Rightarrow t \leq 2s$,
(6) $a + b + r + s + t = 0 \Rightarrow 2a + 2s + t \leq 0$,
(7) $a = b \Rightarrow |r| \leq |s|;\ b = c \Rightarrow |s| \leq |t|$.

By a *primitive* form, we mean one whose coefficients are coprime. Since the theta series associated to a nonprimitive form can be viewed as the *lifts* (see Section 2.4.2 below) of the primitive forms, we may assume that all quadratic forms are primitive.

Given a quadratic form, it is not immediately obvious how to apply Theorem 2.3 since it is not clear *a priori* how to find the level of the corresponding theta series. However, in the case of binary forms, it is quite easy. Let $Q(x, y) = ax^2 + bxy + cy^2$. Then the matrix associated to Q is

$$A_Q = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

Of course,

$$A_Q^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 2c & -b \\ -b & 2a \end{bmatrix},$$

and the level is $\det(A) = 4ac - b^2$, which is $-d$, where d is the discriminant of Q.

In [Leh92], Larry Lehman explicitly shows how to compute the level of a ternary quadratic form and on his webpage even describes an algorithm for writing down all reduced ternary forms of a given level and discriminant. Let

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$$

be a primitive ternary quadratic form. Note that

$$A_Q = \begin{bmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{bmatrix}$$

is the matrix for Q. Then the discriminant of Q is defined to be $\Delta = \dfrac{\det(A)}{2} = 4abc + rst - ar^2 - bs^2 - ct^2$. Let $C_{i,j}$ be the i, j cofactor of A_Q and define the *divisor* of Q to be $m = \gcd(C_{1,1}, C_{2,2}, C_{3,3}, 2C_{2,3}, 2C_{1,3}, 2C_{1,2})$. Then the level of Q is $N = 4\Delta/m$. For proof, see [Leh92].

2.3.1. *Bounds on the Number of Coefficients.* Now we have a way to write modular forms of weight $k/2$ in terms of their Fourier expansions. But to use these forms for computation, we would like to uniquely identify a modular form with just finitely many coefficients.

Theorem 4. [Fre94, Prop 1.1] *Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$, $g(z) = \sum_{n=0}^{\infty} b_n q^n \in M_k(N, \chi)$. If $a_n = b_n$ for $0 \le n \le \frac{kN}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right)$, then $f = g$.*

We can apply this theorem to half integral weight modular forms by noticing that a half integral weight form multiplied by a classical theta series produces an integral weight form, where we can use the above theorem.

Corollary 2. *Let k be odd and $4|N$. Suppose that $f, g \in M_{k/2}(N, \chi)$ such that $f(z) = \sum_{n=0}^{\infty} a_n q^n$, $g(z) = \sum_{n=0}^{\infty} b_n q^n$. Also suppose that $a_n = b_n$ for $0 \le n \le \frac{(k+1)N}{24} \prod_{p|N} \left(1 + \frac{1}{p}\right)$. Then $f = g$.*

Proof. Let $k \in \mathbb{Z}$, k odd, and $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N, \chi)$. Recall that $\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{1/2}(4, \chi_0)$.

Then
$$f(z)\,\theta(z) = \sum_{n \ge 0, m \in \mathbb{Z}} a_n q^{n+m^2} \in M_{\frac{k+1}{2}}(N, \psi).$$
By Theorem 2, we have the following minimum bound on the number of coefficients of $f(z)\,\theta(z)$ that we must calculate:
$$\frac{\left(\frac{k+1}{2}\right) N}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right) = \frac{(k+1)\,N}{24} \prod_{p|N} \left(1 + \frac{1}{p}\right)$$
Hence we must compute a_n for $0 \le n \le \frac{(k+1)N}{24} \prod_{p|N} \left(1 + \frac{1}{p}\right)$. ☐

We can now compute the $q-$series for a theta series by evaluating the quadratic form Q over $\vec{x} \in \mathbb{Z}^k$, where $k = 2$ or 3. After calculating the bound B for the exponents of q in the Fourier expansion, we can find bounds for the components of $\vec{x} \in \mathbb{Z}^k$ such that $Q(\vec{x}) \le B$. These bounds are used to limit the number of $\vec{x}$ we must test to find the coefficients of the Fourier series.

Theorem 5. *Let $Q(x, y)$ be a binary quadratic form. If $Q(x_0, y_0) \le B$, then $|x_0| \le \sqrt{\frac{4cB}{4ac-b^2}}$ and $|y_0| \le \sqrt{\frac{4aB}{4ac-b^2}}$.*

Proof. Suppose $Q(x_0, y_0) = ax_0^2 + bx_0y_0 + cy_0^2 \le B$. Rearranging and completeing the square, we can see
$$a\left(x_0^2 + \frac{by_0}{a}x_0 + \frac{(by_0)^2}{4a^2}\right) - \frac{(by_0)^2}{4a} + cy_0^2 = a\left(x_0 + \frac{by_0}{2a}\right)^2 + \frac{4ac-b}{4a}y_0^2 \le B.$$
Since $a\left(x_0 + \frac{by_0}{2a}\right)^2$ is always positive, this implies $\frac{4ac-b}{4a}y_0^2 \le B$, and by solving for y_0 we arrive at the above bound. An identical process can be used to solve for the x_0 bound. ☐

Theorem 6. *Let $Q(x, y, z) \le B$ be a ternary quadratic form. If $Q(x_0, y_0, z_0) \le B$, then*
$$|x_0| \le \sqrt{\frac{(4bc - r^2)\,B}{4abc + rst - ar^2 - bs^2 - ct^2}},$$
$$|y_0| \le \sqrt{\frac{(4ac - s^2)\,B}{4abc + rst - ar^2 - bs^2 - ct^2}},$$

$$|z_0| \leq \sqrt{\frac{(4ab - t^2)\, B}{4abc + rst - ar^2 - bs^2 - ct^2}}.$$

Proof. This case is similar to the binary case, except we have to complete the square twice. □

2.4. **Operations on Modular Forms.** In this section we dicuss ways to create forms from the theta series we have made.

2.4.1. *Multiplication of Theta Series.* One way to create new modular forms is by multiplying two forms together. The following three facts explain the results of multiplying two modular forms together.

Fact 4. Let $f \in M_{k_1}(N, \chi)$ and $g(z) \in M_{k_2}(N, \psi)$. Then $fg \in M_{k_1+k_2}([M, N], \chi\psi)$ where $[M, N]$ is the least common multiple of M and N.

Proof. This proof is very similar to (and easier than) the proofs of the following two facts. □

Fact 5. Let $k_1, k_2 \in \mathbb{Z}$, k_1, k_2 odd. Also let $f \in M_{k_1/2}(N, \chi)$, $g \in M_{k_2/2}(M, \psi)$. Define χ_{-1} to be the Kronecker symbol $\left(\frac{-1}{d}\right)$.

1. If $k_1 \not\equiv k_2 \pmod 4$, then $fg \in M_{(k_1+k_2)/2}([M, N], \chi\psi)$
2. If $k_1 \equiv k_2 \pmod 4$, then $fg \in M_{(k_1+k_2)/2}([M, N], \chi\psi\chi_{-1})$

Proof. Let $\gamma \in \Gamma_0([M, N])$. First take $c \neq 0$. Then

$$\begin{aligned} f(\gamma z)\, g(\gamma z) &= \chi\psi(d)\, \chi_c^2(d)\, \epsilon_d^{-(k_1+k_2)} \left(\sqrt{cz+d}\right)^{(k_1+k_2)} f(z)\, g(z) \\ &= \chi\psi(d)\, \epsilon_d^{-(k_1+k_2)} (cz+d)^{(k_1+k_2)/2} f(z)\, g(z), \end{aligned}$$

since χ_c is a quadratic character and therefore squares to 1. Since k_1 and k_2 are odd, fg is an integer weight form. Now look at the character.

We know ϵ_d is a forth root of unity, so we only have to look at its exponent modulo 4. When $k_1 \not\equiv k_2 \pmod 4$ then $k_1 + k_2 \equiv 0 \pmod 4$ and $\epsilon_d^{-(k_1+k_2)} = 1$. Then the character for fg is just $\chi\psi$. When $k_1 \equiv k_2 \pmod 4$, then $k_1 + k_2 \equiv 2 \pmod 4$ and $\epsilon_d^{-(k_1+k_2)} = \left(\frac{-1}{d}\right)$. Then the character is $\chi\psi\chi_{-1}$.

The last case is when $c = 0$. In this case $\chi_c(d) = \left(\frac{0}{d}\right) = 0$ for $d \neq \pm 1$. But since $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$ we have $a = d = 1$ or $a = d = -1$. Therefore $\chi_c(d) = \left(\frac{0}{\pm 1}\right) = 1$ and fg is an integer weight form with character as above. □

Fact 6. Let $k_1, k_2 \in \mathbb{Z}$, k_2 odd. Also let $f \in M_{k_1}(N, \chi)$, $g \in M_{k_2/2}(M, \psi)$. Recall that χ_{-1} is the Kronecker symbol $\left(\frac{-1}{d}\right)$.

1. If k_1 is even, then $fg \in M_{(2k_1+k_2)/2}([M, N], \chi\psi)$
2. If k_1 is odd, then $fg \in M_{(2k_1+k_2)/2}([M, N], \chi\psi\chi_{-1})$

Proof. Again let $\gamma \in \Gamma_0([M, N])$ and take $c \neq 0$. Then

$$f(\gamma z)\, g(\gamma z) = \chi\psi(d)\, \chi_c(d)\, \epsilon_d^{-k_2} (cz+d)^{(2k_1+k_2)/2} f(z)\, g(z).$$

Notice fg is a half-integer weight form. We consider the case where k_1 is even separately from the case where k_1 is odd.

First look at k_1 even. Then $k_2 \equiv 2k_1+k_2 \pmod 4$, so $\epsilon_d^{-k_2} = \epsilon_d^{-(2k_1+k_2)}$, and $fg \in M_{(2k_1+k_2)/2}([M, N], \chi\psi)$. When k_1 is odd, then ψ is odd and $k_2 \equiv 2k_1 + k_2 + 2 \pmod 4$. Then

$$fg = \left(\chi\psi(d)\, \epsilon_d^{-2}\right) \chi_c(d)\, \epsilon_d^{-(2k_1+k_2)} (cz+d)^{(2k_1+k_2)/2} f(z)\, g(z),$$

but $\epsilon^{-2} = \left(\frac{-1}{d}\right)$, so $fg \in M_{(2k_1+k_2)/2}\left([M,N], \chi\psi\chi_{-1}\right)$.

Now all that is left is the case where $c = 0$. Again we have $a = d = 1$ or $a = d = -1$, and therefore $\chi_c(d) = \left(\frac{0}{\pm 1}\right) = 1$. So fg is a half-integral weight form with character as above. □

The previous fact tells us that multiplying a weight 1 form with a weight 1/2 form will return a weight 3/2 form. Since we are interested in constructing forms of weight 3/2, it seems only natural that we will make great use of this technique later on.

2.4.2. *Lifting.* In our calculations, we consider one type of lifting on weight 1 forms and two different types of lifting on weight 3/2 forms to create forms.

Fact 7. Let $k \in \frac{1}{2}\mathbb{Z}$. Then $M_k(N,\chi) \subseteq M_k(mN,\chi)$ for any $m \in \mathbb{Z}^+$.

Proof. This is a direct consequence from the fact that $\Gamma_0(mN) \leq \Gamma_0(N)$. □

We refer to the action of placing a form f from the space $M_{k/2}(N,\chi)$ into the space $M_{k/2}(mN,\chi)$ as including f by m and denote this $I_m(f)$. This applies identically to integer and half-integer weight forms.

The action we call V-lifting differs depending on whether the weight is an integer or a half-integer.

Fact 8. Let $k \in \mathbb{Z}$. For a modular form $f(z) \in M_k(N,\chi)$, the form $g(z) = f(Mz) \in M_k(MN,\chi)$.

Proof. See [Kob93, pp. 127-129]. □

Fact 9. Let $\lambda \in \mathbb{Z}$. For $f(z) \in M_{\lambda+\frac{1}{2}}(4N,\chi)$, the form $g(z) = f(dz) \in M_{\lambda+\frac{1}{2}}\left(4Nd, \left(\frac{4d}{\bullet}\right)\chi\right)$.

Proof. See [Ono04, pp. 50] □

We use the notation $V_d(f)$ to denote the V-lift of $f(z)$ by d. V-lifting can be thought of as "spreading out" the coefficients in the q-series of f by a factor of d.

2.4.3. *Twisting.* The action of twisting a modular form by a Dirichlet character is another way to create forms. For modular forms of integer or half-integer weight, the twisting process is as follows.

Fact 10. Let $k \in \frac{1}{2}\mathbb{Z}$ and let $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(N,\chi)$. If ψ is a primitive character of conductor m, we define the $\psi-$*twist* of $f(z)$ to be $f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a(n) q^n$ and we can conclude that $f_\psi(z) \in M_k(Nm^2, \chi\psi^2)$.

Proof. See [Ono04, pp. 23]. □

We denote the act of twisting a form $f(z)$ by a character ψ as $T_\psi(f)$.

2.4.4. *Hecke Operators.* We present the action of Hecke operators on integer weight forms as described in [Ono04, pp. 21-22].

Fact 11. For $k \in \mathbb{Z}$ and p prime, if $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(N,\chi)$, then the action of the Hecke operator $T_{p,k,\chi}$ of $f(z)$ is defined by

$$f(z)\,|T_{p,k,\chi} = \sum_{n=1}^{\infty} \left(a(pn) + \chi(p)\, p^{k-1} a(n/p)\right) q^n.$$

As convention, we say that if $p \nmid n$, then we agree that $a(n/p) = 0$. We conclude that $f(z)\,|T_{m,k,\chi} \in M_k(N,\chi)$.

Proof. See [Ono04, pp. 21-22]. □

The action of Hecke operators on half-integral weight forms as described in [Ono04, pp. 49-50] is similar.

Fact 12. For $\lambda \in \mathbb{Z}$, if $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_{\lambda+\frac{1}{2}}(4N, \chi)$, then the half-integral weight Hecke operator $T_{p^2,\lambda,\chi}$ is defined by

$$f(z) | T_{p^2,\lambda,\chi} = \sum_{n=0}^{\infty} \left(a\left(p^2 n\right) + \chi^*(p) \left(\frac{n}{p}\right) p^{\lambda-1} a(n) + \chi^*\left(p^2\right) p^{2\lambda-1} a\left(n/p^2\right) \right) q^n.$$

If p is prime, then $f(z) | T_{p^2,\lambda,\chi} \in M_{\lambda+\frac{1}{2}}(4N, \chi)$.

Proof. See [Ono04, pp. 49-50]. ☐

Here χ^* is the Dirichlet character given by $\chi^*(n) = \left(\frac{(-1)^{\lambda}}{n}\right) \chi(n)$, and if $p^2 \nmid n$ then $a(n/p^2) = 0$.

For both integer and half-integral weights, if $f(z)$ is a cusp form then $f(z) | T_{m,k,\chi}$ is as well.

Definition 4. A space $M_{3/2}(N_1, d_1)$ is a *feeder space* of $M_{3/2}(N, d)$ if for all $f(z) \in M_{3/2}(N_1, d_1)$ there is some n or χ such that $I_n(f)$, $V_n(f)$, or $T_\chi(f)$ is in $M_{3/2}(N, d)$.

3. Computation

We compute spaces of weight 3/2 modular forms using Sage. In our computations, we only consider those spaces $M_{3/2}(N, \chi)$ such that χ has order dividing 2. Note that a quadratic character χ is equivalent to a unique Kronecker character $\left(\frac{d}{\cdot}\right)$ where d is squarefree (see [Ste07] for an algorithm for converting between Dirichlet and Kronecker characters in Sage). Thus in our computations we use the shorthand $M_k(N, d)$ to refer to the space $M_k\left(N, \left(\frac{d}{\cdot}\right)\right)$. Throughout this section, let sqf(n) refer to the squarefree part of n.

There are two approaches to constructing spaces of weight 3/2 forms. Roughly speaking, these are "bottom-up" and "top-down". In the bottom-up approach, we set a maximum level, referred to henceforth as MAXLEVEL, for the spaces to be created, and proceed to create all spaces of modular forms of level less than MAXLEVEL. We begin at weight 1/2, and construct all lifts and twists of the theta function that keep the level below MAXLEVEL. We multiply these together to get weight 1 forms. At weight 1, we add in modular forms from binary quadratic forms, and then take lifts and twists of all forms created thus far. To create forms of weight 3/2, we multiply all weight 1 forms against all weight 1/2 forms, then add in modular forms from ternary quadratic forms. We then take lifts and twists of the forms at weight 3/2.

The top-down approach allows us to construct a single space at a time. For this, we employ algorithms which enumerate all the spaces which "feed in" to our desired space via lifting, twisting, or multiplication (the feeder spaces of Definition 4 above), and then recursively compute these spaces in order to map the forms created into our space. Before describing the algorithms in detail, we first need a few small theorems.

Proposition 1. *Including at weight 1 will not create any forms that are not already created by Including at weight 3/2.*

Proof. Let $f(z) \in M_1(N, \chi)$. Then $I_m(f)$ places $f(z) \in M_1(mN_1, \chi)$ for all $m \in \mathbb{Z}$. Now notice that the only way to create a weight 3/2 form from $f(z)$ in our algorithm is by multiplying by a weight 1/2 form. Let $g(z) \in M_{1/2}(N_2, \psi)$. Then $f(z) g(z) \in M_{3/2}([mN_1, N_2], \chi\psi\chi_{-1})$. But if we first multiply, $f(z) g(z) \in M_{3/2}([N_1, N_2], \chi\psi\chi_{-1})$, since $[N_1, N_2] | [mN_1, N_2]$, we can say for some $n \in \mathbb{Z}$, $I_n(fg)$ places $f(z) g(z) \in M_{3/2}([mN_1, N_2], \chi\psi\chi_{-1})$. Therefore any form created by including a weight 1 form will also be created by including the weight 3/2 forms. ☐

Proposition 2. *Given $f \in M_{3/2}(N,\chi)$, we only need to consider V-lifting and including by primes $p \leq MAXLEVEL/N$.*

Proof. Look first at inclusion-lifting. Let $f(z) \in M_{3/2}(N,\chi)$. Apply an include-lift to the form: $I_p(f)$ places $f(z) \in M_{3/2}(pN,\chi)$. Applying the include-lift again, $I_q(I_p(f))$ places $f(z) \in M_{3/2}((pq)N,\chi)$. Notice that $I_{pq}(f)$ also places $f(z) \in M_{3/2}((pq)N,\chi)$, so $I_q(I_p(f))$ is the same as $I_{pq}(f)$. This means that an include-lift by a composite number m and a series of include-lifts by the prime factors of m are equivalent, so we only need to include-lift by primes.

Now look at V-lifting. Again let $f(z) \in M_{3/2}(N,\chi)$. By a similar argument as above, since $V_p(V_q(f)) = V_{pq}(f)$, V-lifting by primes is sufficient.

□

3.1. **Construction of All Spaces Below MAXLEVEL.** In this section we describe the algorithm for the computation of all spaces $M_{3/2}(N,d)$ for $N \leq$ MAXLEVEL.

Algorithm 1. Construction of all spaces $M_{3/2}(N,d)$ for $N \leq$ MAXLEVEL.

(1) Construct all theta series $\theta_{\psi,t}$ for primitive even characters ψ and positive integers t.
   (a) Iterate t from 1 to MAXLEVEL/4.
   (b) For each t, generate all primitive even quadratic characters ψ with conductor r, where r ranges from 1 to $\sqrt{\frac{\text{MAXLEVEL}}{4t}}$
   (c) Create the theta-series $\theta_{\psi,t}$.

(2) Multiply all pairs of created theta series of levels N_1 and N_2 with $[N_1,N_2] \leq$ MAXLEVEL to create weight one forms.

(3) Create weight 1 forms from binary quadratic forms.
   (a) For each level $N \leq$ MAXLEVEL, compute all primitive reduced positive definite binary quadratic forms (see Algorithm 3).
   (b) For each binary quadratic form of level N, compute the associated weight 1 modular form of level N and character $\left(\frac{-N}{\cdot}\right)$. See Algorithm 6 below.

(4) For all weight 1 forms f of level N and all primes $p \leq$ MAXLEVEL/N, apply $V_p(f)$ the V-lift operator.

(5) For all weight 1 forms f of level N and character $\left(\frac{d}{\cdot}\right)$ and all primitive nontrivial characters χ of conductor $t \leq \sqrt{\frac{\text{MAXLEVEL}}{N}}$ and order dividing 4, apply the twist operator $T_\chi(f)$.

(6) For all N_1, N_2 such that $[N_1,N_2] \leq$ MAXLEVEL, multiply all forms of weight 1/2 and level N_1 against all forms of weight 1 and level N_2 to create forms of weight 3/2.

(7) Create weight 3/2 forms from ternary quadratic forms.
   (a) Compute all primitive reduced positive definite ternary quadratic forms of level $N \leq$ MAXLEVEL and any discriminant using Algorithm 5.
   (b) For each ternary quadratic form of level N and discriminant d, compute the associated weight 3/2 modular forms of level N and character $\left(\frac{d}{\cdot}\right)$. See Algorithm 7 below.

(8) For all weight 3/2 forms f of level N and all primes $p \leq$ MAXLEVEL/N, apply $I_p(f)$ the inclusion-lift operator and $V_p(f)$ the V-lift operator.

(9) For all weight 3/2 forms f of level N and character $\left(\frac{d}{\cdot}\right)$ and all primitive nontrivial characters χ of conductor $t \leq \sqrt{\frac{\text{MAXLEVEL}}{N}}$ and order dividing 4, apply the twist operator $T_\chi(f)$.

3.2. **Construction of $M_k(N,d)$.** In this section we describe the procedure to construct the specific space $M_k(N,d)$. The algorithm proceeds by recursion: first one determines all spaces $M_j(N_1,d_1)$

which feed into $M_k(N,d)$ via multiplication, lifting, or twisting, then one computes these spaces and performs the appropriate operations to construct forms in $M_k(N,d)$.

Algorithm 2. Construction of $M_k(N,d)$.

(1) If $k = 1/2$, it suffices to compute the basis for $M_{1/2}(N,d)$ given in [SS77] directly. See Algorithm 12 below.
(2) Otherwise (for $k = 1$ or $k = 3/2$), create weight k forms from binary or ternary quadratic forms.
  (a) For each level N, compute all primitive reduced positive definite $2k$-ary quadratic forms of level $N \leq$ MAXLEVEL and any discriminant. See Algorithm 3 and 5 below.
  (b) For each $2k$-ary quadratic form of level N and discriminant d, compute the associated weight-k modular forms of level N and character $\left(\frac{d}{\cdot}\right)$ using Algorithm 6 or 7.
(3) If $k = 3/2$, using Algorithm 8, enumerate all spaces $M_{3/2}(N_1,d_1)$ such that I_p maps $M_{3/2}(N_1,d_1)$ to $M_{3/2}(N,d)$ for some prime p. Apply the appropriate I_p.
(4) Using Algorithm 9, enumerate all spaces $M_k(N_1,d_1)$ such that V_p maps $M_k(N_1,d_1)$ to $M_k(N,d)$ for some prime p. Apply the appropriate V_p.
(5) Using Algorithm 10, enumerate all spaces $M_k(N_1,d_1)$ such that T_χ maps $M_k(N_1,d_1)$ to $M_k(N,d)$ for some primitive character χ of order dividing 4. Apply the appropriate T_χ.
(6) Using Algorithm 11, compute all spaces $M_{k_1}(N_1,d_1)$ and $M_{k_2}(N_2,d_2)$ such that for $f \in M_{k_1}(N_1,d_1)$ and $g \in M_{k_2}(N_2,d_2)$, $fg \in M_k(N,d)$. Perform all multiplications.

3.3. **Computing Quadratic Forms.** In this section we describe the algorithms used in generating reduced binary and ternary quadratic forms. We must first establish some preliminary results which make the problem computationally tractable. Here and throughout, the notation $\lfloor x \rfloor$ will denote the floor function: the greatest integer less than or equal to x.

The following result establishes that we need only consider finitely many possibilities for the coefficients a,b,c on a reduced binary quadratic form of a given level.

Proposition 3. *Let $Q(x,y) = ax^2 + bxy + cy^2$ be a reduced positive definite binary quadratic form with level N. Then $a \leq \sqrt{N/3}$.*

Proof. The form Q has discriminant $D = b^2 - 4ac$. Since Q is reduced, we have that $|b| \leq a \leq c$. It follows that $b^2 \leq a^2$. Recalling that $a \leq c$ as well, we have:

$$D = b^2 - 4ac \leq a^2 - 4a^2 = -3a^2$$

For binary forms, since $N = -D$, it follows that $a^2 \leq \sqrt{\frac{N}{3}}$. ☐

These results allow for a simple algorithm to calculate all primitive, reduced, positive definite binary quadratic forms of level N.

Algorithm 3. Determination of all primitive positive-definite reduced binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ of level N.

(1) Let a range from 1 to $\lfloor \sqrt{N/3} \rfloor$.
(2) Given a, let b range from $-a$ to a.
(3) Given a and b, check if $c = \frac{b^2+N}{4a}$ is an integer and that $c \geq a$.
(4) Check that $\gcd(a,b,c) = 1$.
(5) Given a,b,c integers, check if $a = c$ or $|b| = a$.
(6) If neither conditions are met, the form $Q(x,y) = ax^2 + bxy + cy^2$ is primitive, reduced and positive definite of level N.

(7) If one of the conditions in (4) is met, check that $b \geq 0$. If so, then $Q(x,y) = ax^2 + bxy + cy^2$ is primitive, reduced and positive definite of level N.

For ternary forms, the relationship between level and discriminant is somewhat more complicated. In the ternary-form case, there are multiple discriminants which may be associated to a given level. Our algorithm proceeds in two stages. First, using some simple divisibility criteria, we can consider only finitely many possible discriminants for a given level. At each of these discriminants, we employ an algorithm of Lehman for enumerating all reduced positive definite ternary quadratic forms; c.f. [Leh92]. Our first result establishes some restrictions on the values of N and D.

Proposition 4. *Let $Q(x,y,z)$ denote a ternary quadratic form of level N and discriminant D. Let $m = \frac{4D}{N}$ and $\mu = \frac{4N}{m}$. Then the following hold:*

(1) $N \equiv 0 \pmod 4$
(2) $m, \mu \in \mathbb{Z}^+$
(3) $N/4 \leq D \leq N^2$
(4) $D \equiv 0 \pmod{N/4}$

Proof. For proofs of (1) and (2), see Lehman [Leh92].
(3): Since $m = 4D/N \geq 1$, it follows that $D \geq N/4$. Also, we have that $\mu = 4N/m = N^2/D \geq 1$, implying $D \leq N^2$.
(4) This follows immediately from the condition that $m = \frac{4D}{N}$ is an integer. □

Items (3) and (4) in Proposition 4 greatly narrow the search space of possible discriminants for reduced ternary quadratic forms of a given level. In conjunction with the algorithm described by Lehman on his webpage to enumerate reduced positive definite ternary quadratic forms of level N and discriminant D, one can write down all ternary quadratic forms of a given level.

Algorithm 4 (Lehman [Leh92])**.** Determination of all primitive positive definite reduced ternary quadratic forms

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$$

of level N and discriminant D.

(1) Let D range from $N/4$ to N^2, subject to $D \equiv 0 \pmod{N/4}$.
(2) Given D, apply Algorithm 4 to produce all reduced positive definite ternary quadratic forms of level N and discriminant D.

Algorithm 5 (Ternary Quadratic Forms)**.** Determination of all positive definite reduced ternary quadratic forms of level N.

(1) Let a range from 1 to $\lfloor \sqrt[3]{D/2} \rfloor$.
(2) Given a, let b range from a to $\lfloor \sqrt{D/(2a)} \rfloor$.
(3) Given a, b, let r range from $-b$ to b.
(4) Given a, b, r, let s range from $-a$ to a.
(5) Given a, b, r, s, let t range from $-a$ to a.
(6) Check if $c = (D - rst + ar^2 + bs^2) / (4ab - t^2)$ is an integer.
(7) Check that $\gcd(a, b, c, r, s, t) = 1$
(8) Check that $a + b + c + r + s + t \geq 0$.
(9) Check that the form $ax^2 + by^2 + cz^2 + ryz + sxz + txy$ is reduced.

(10) Check that the level of the form is N.

3.4. **Computing q-Series.** Recall that by Theorem 4 and Corollary 2, for computational purposes, it suffices to consider only finitely many coefficients on the q-series of a modular form f. We next present a procedure to compute finitely many coefficients of the modular form associated to a quadratic form Q. Per Theorems 5 and 6 we need only compute Q at finitely many points.

Algorithm 6. Construction of B coefficients of the modular form f_Q associated to the primitive positive-definite reduced binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ of level N. Note that level N will give f_Q character $\left(\frac{-N}{\cdot}\right)$; in our notation, $f \in M_1(N, \text{sqf}(-N))$.

(1) For all $x \in \mathbb{Z}$ such that $|x| \leq \sqrt{\left(\frac{4cB}{4ac-b^2}\right)}$ and all $y \in \mathbb{Z}$ such that $|y| \leq \sqrt{\left(\frac{4aB}{4ac-b^2}\right)}$, compute $Q(x, y) = n$.
(2) If $n \leq B$, increase the coefficient on q^n in the Fourier series for f_Q by 1.

Algorithm 7. Construction of the first B coefficients of the modular form f_Q associated to the primitive, positive-definite, reduced ternary quadratic form $Q(x, y, z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ of level N and discriminant d. Note that $f_Q \in M_{3/2}(N, \text{sqf}(d))$.

(1) For all $x \in \mathbb{Z}$ such that:
$$|x| \leq \sqrt{\frac{(4bc - r^2)\, B}{(4abc + rst - ar^2 - bs^2 - ct^2)}}$$
and for all $y \in \mathbb{Z}$ such that:
$$|y| \leq \sqrt{\frac{(4ac - s^2)\, B}{(4abc + rst - ar^2 - bs^2 - ct^2)}}$$
and for all $z \in \mathbb{Z}$ such that:
$$|z| \leq \sqrt{\frac{(4ab - t^2)\, B}{(4abc + rst - ar^2 - bs^2 - ct^2)}}$$
compute $Q(x, y, z) = n$.
(2) If $n \leq B$, increase the coefficient on q^n in the Fourier series for f_Q by 1.

3.5. **Determination of Divisor Spaces.** In this section we present algorithms to enumerate all feeder spaces $M_{k_1}(N_1, d_1)$. We also present an algorithm to compute the set $\Omega\left(N, \left(\frac{d}{\cdot}\right)\right)$ which will give a basis for the space $M_{1/2}(N, d)$ using the result of Serre and Stark. Recall that per Proposition 1, we need only consider lifts by prime indices. Throughout this section, we adopt the convention that an arbitrary k is taken to be either an integer or a half-integer.

In the following sections we present divisibility criteria for each of our maps. The results of this paragraph will hold for all such maps. By definition, a given space $M_k(N, d)$ must have $r \mid N$, where r is the conductor of $\left(\frac{d}{\cdot}\right)$. Also, for k a half-integer, we must have that $4 \mid N$ for the space $M_k(N, d)$ to be nonempty.

3.5.1. *Inclusion Mapping.* Let p be prime. The inclusion mapping I_p sends $M_k(N_1, d)$ into $M_k(pN_1, d)$. Suppose I_p maps $M_k(N_1, d)$ into $M_k(N, d)$. Then clearly $N_1 \mid N$. Note that I_p has no effect on the character of the space.

Algorithm 8. Enumeration of all spaces $M_k(N_1, d)$ that map under I_p to $M_k(N, d)$ for some prime p.

(1) For all proper divisors N_1 of N check that N/N_1 is prime. If $k = 3/2$, also check that $N_1 \equiv 0 \pmod 4$.
(2) Given N_1, check that $d \mid N_1$. If $k = 3/2$, also check that $d \mid N_1/4$.

3.5.2. *V-Lifts.* Let p be prime. For weight $k = 1$, V_p sends $M_1(N_1, d_1)$ to $M_1(pN_1, d_1)$. At weight $k = 3/2$, V_p sends $M_{3/2}(N_1, d_1)$ to $M_{3/2}(pN_1, \mathrm{sqf}(pd_1))$. Suppose the map V_p sends $M_k(N_1, d_1)$ into $M_k(N, d)$. Then the following hold:

(1) For $k = 1$, $d_1 = d$, $d \mid N_1$, $N_1 \mid N$ and $N/N_1 = p$ is prime.
(2) For $k = 3/2$, $N_1 \mid N$, $N_1 \equiv 0 \pmod 4$, $d_1 \mid N_1/4$ and $\mathrm{sqf}(pd_1) = d$

Algorithm 9. Enumeration of all spaces $M_k(N_1, d_1)$ that map under V_p to $M_k(N, d)$ for some prime p.

(1) If $k = 1$, since I_p and V_p map to the same space, simply apply Algorithm 8.
(2) For $k = 3/2$, iterate over all proper divisors N_1 of N such that $N_1 \equiv 0 \pmod 4$ and such that N/N_1 is prime.
(3) Given N_1, iterate over all divisors d_1 of $N_1/4$
(4) Given N_1 and d_1, check that d_1 is squarefree and that $\mathrm{sqf}(d_1 N/N_1) = d$.

3.5.3. *Twisting.* Let χ be a primitive character of modulus m and order dividing 4. Then χ^2 is a Kronecker character $\left(\frac{t}{\cdot}\right)$ for some t. The map T_χ sends $M_k(N_1, d_1)$ to $M_k(m^2 N_1, \mathrm{sqf}(td_1))$ Suppose $M_k(N_1, d_1)$ maps to $M_k(N, d)$ under the map T_χ. Then the following hold:

(1) $N_1 \mid N$ and $N/N_1 = m^2$. If $k = 3/2$, then also $N_1 \equiv 0 \pmod 4$.
(2) $d_1 \mid N_1$. If $k = 3/2$, then also $d_1 \mid N_1/4$.
(3) $\mathrm{sqf}(d_1 t) = d$.

Algorithm 10. Enumeration of all spaces $M_k(N_1, d_1)$ that map under T_χ to $M_k(N, d)$ for some primitive character χ of order dividing 4.

(1) For all divisors N_1 of N such that $|N_1| < N$, check that $N/N_1 = m^2$ is a square. If $k = 3/2$, also check that $N_1 \equiv 0 \pmod 4$.
(2) For all proper divisors d_1 of N_1, check that d_1 is squarefree. For $k = 3/2$, it suffices to consider all squarefree divisors of $N_1/4$.
(3) Let χ^2 be denoted by the Kronecker character $\left(\frac{t}{\cdot}\right)$. For all primitive characters of conductor m and order dividing 4, check that $\mathrm{sqf}(d_1 t) = d$.

3.5.4. *Multiplication.* Let $M_{k_1}(N_1, d_1)$ and $M_{k_2}(N_2, d_2)$ be spaces of modular forms. We consider the cases $k_1 = k_2 = 1/2$ and $k_1 = 1/2, k_2 = 1$. For $k_1 = k_2 = 1/2$, take $f \in M_{1/2}(N_1, d_1)$ and $g \in M_{1/2}(N_2, d_2)$. Then:

$$fg \in M_1([N_1, N_2], \mathrm{sqf}(-d_1 d_2))$$

For $k_1 = 1/2$, $k_2 = 1$, take $f \in M_{1/2}(N_1, d_1)$ and $g \in M_1(N_2, d_2)$. Then:

$$fg \in M_{3/2}([N_1, N_2], \mathrm{sqf}(-d_1 d_2))$$

Suppose the product of forms in $M_{k_1}(N_1, d_1)$ and $M_{k_2}(N_1, d_1)$ yields forms in $M_{k_1+k_2}(N, d)$. Then the following hold:

(1) $[N_1, N_2] = N$
(2) $\mathrm{sqf}(-d_1 d_2) = d$

Algorithm 11. Enumeration of all pairs of spaces $M_{k_1}(N_1, d_1)$ and $M_{k_2}(N_2, d_2)$ where the product yields forms in $M_k(N, d)$.

- For $k = 1$, set $k_1 = k_2 = 1/2$:

(1) Iterate through all divisors N_1 of N such that $N_1 \equiv 0 \pmod 4$.
(2) Given N_1, iterate through all divisors N_2 of N such that $N_2 \equiv 0 \pmod 4$ and such that $N_2 \geq N_1$.
(3) Given N_1 and N_2, iterate through all squarefree divisors d_1 of $N_1/4$.
(4) Given N_1, N_2 and d_1, iterate through all squarefree divisors d_2 of $N_2/4$.
(5) If $N_1 = N_2$, check that $d_2 \geq d_1$.
(6) Given N_1, N_2, d_1 and d_2, check that $\mathrm{sqf}(d_1 d_2) = -d$.

- For $k = 3/2$, set $k_1 = 1/2$ and $k_2 = 1$:
  (1) Iterate through all divisors N_1 of N such that $N_1 \equiv 0 \pmod 4$.
  (2) Given N_1, iterate through all divisors N_2 of N. Note that since N_2 is the level of an integral-weight space, there will be nonempty spaces even for N_2 not divisible by 4.
  (3) Given N_1 and N_2, iterate through all squarefree divisors d_1 of $N_1/4$.
  (4) Given N_1, N_2 and d_1, iterate through all d_2 such that $-d_2$ is a positive squarefree divisor of N_2. Let r be the conductor of $\left(\frac{d_2}{\cdot}\right)$. Check that $r \mid N_2$.
  (5) Given N_1, N_2, d_1 and d_2, check that $\mathrm{sqf}(-d_1 d_2) = d$

Algorithm 12. Construction of $\Omega\left(N, \left(\frac{d}{\cdot}\right)\right)$

(1) Iterate through all r such that $1 \leq r \leq \lfloor\sqrt{N/4}\rfloor$ and $4r^2 \mid N$.
(2) For all primitive even characters ψ of order 2 and conductor r, determine the Kronecker character $\left(\frac{a}{\cdot}\right)$ associated to ψ.
(3) Given ψ, r and a, set $s = \mathrm{sqf}(ad)$.
(4) For $1 \leq t$ such that $4r^2 t \mid N$, check that $s = \mathrm{sqf}(t)$.
(5) Return the pair (ψ, t).

3.6. **Data.** In this section we discuss the results of our computation and introduce some conjectures. Let $A_{3/2}(N,d)$ denote the space generated by Algorithm 1 or Algorithm 2. We have computed $A_{3/2}(N,d)$ for all spaces with level $N \leq 600$, extending the tables of Smith, who previously considered all $N \leq 200$. In addition, let $B_{3/2}(N,d)$ denote the subspace of $A_{3/2}(N,d)$ generated just from lifts of ternary-form-derived modular forms (so omitting any twisting or multiplication). We also present computations of the dimension of $B_{3/2}(N,d)$ for $N \leq 600$, with data up to $N \leq 1000$ available upon request. In those cases where $\dim\left(A_{3/2}(N,d)\right) > \dim\left(B_{3/2}(N,d)\right)$, we introduce and compute the space $C_{3/2}(N,d)$ which is the space generated by all lifts and twists of forms at weight three-halves (so omitting multiplication).

We find that for spaces with level $N \leq 96$, our algorithm does generate a basis. The first space for which the algorithm comes up short is at level 100: the space $M_{3/2}(100,5)$ has dimension 12, whereas $\dim\left(A_{3/2}(100,5)\right) = 11$. Further computation shows that the size of $A_{3/2}(N,d)$ relative to $M_{3/2}(N,d)$ is governed by the number of divisors of N. That is, the more divisors of N, the more of $M_{3/2}(N,d)$ we generate. This is reasonable, given that more divisors of the level will yield more feeder spaces and so more forms overall.

Observe that in the majority of spaces, $\dim\left(A_{3/2}(N,d)\right) = \dim\left(B_{3/2}(N,d)\right)$. This implies that in most cases, forms derived from twisting and multiplication are in the span of lifts of ternary quadratic forms. We note two exceptions to this: for level N such that $100 \mid N$, twisting does yield linearly-independent forms. For $N = 2^k$ with $k \geq 8$ as well, twisting expands the space. In both these cases, we conjecture that the introduction of quartic characters is responsible for the linearly-independent forms. The first r for which there are quartic characters of conductor r is $r = 5$; this generates forms of level dividing 100. The next such r is 8, which will create forms with level dividing 256. Note that twisting by quartic characters will introduce complex coefficients on the q-series for f; twisting by a quadratic character will yield only real (even integral) coefficients.

As another possible method for generating linearly-independent forms, we implemented Hecke operators on twisted forms. We found that in all tested cases, this did not enlarge our space. However, since applying the Hecke operator $T_{p^2,1,\chi}$ requires increasing the Sturm bound by a factor of p^2, we found that exhaustive investigation proved computationally infeasible. In particular, we would like to investigate the effect of $T_{p^2,1,\chi}$ on spaces with level $4p$; the first space for which this applies (i.e. where $\dim\left(M_{3/2}\left(N,d\right)\right) > \dim\left(A_{3/2}\left(N,d\right)\right)$) has level $N = 116$. Unfortunately, this would require computing over $30,000$ coefficients on the modular forms, which is beyond the capabilities of our present code.

3.7. **Conjectures and Future Work.** Our computations suggest a number of conjectures and questions for which there was not sufficient time to investigate during the REU. First, we would like to classify all spaces that do have a basis of lifts and twists of modular forms from quadratic forms. Extending this, we would like to quantify the size of our generated space. We would like to extend our computations further to determine whether there is a class of levels that always have a basis of our forms, or whether these observations are merely artifacts of the relatively small levels involved. In the further pursuit of a basis for spaces of weight three-half modular forms, we would like to introduce η-quotients, and determine if these provide a way of generating linearly-independent forms.

Our conjecture on quadratic twists has a connection to the theory of quadratic forms. Let Q be a quadratic form, and let $r_Q\left(n\right)$ denote the number of solutions to $Q\left(\vec{x}\right) = 0$. Our conjecture that quadratic twists are always linearly-dependent on untwisted forms is equivalent to a family of identities involving various r_{Q_i}. As an example, we use our computed basis for $M_{3/2}\left(36,1\right)$ to derive some identities involving the following quadratic forms:

$$\begin{aligned} Q_0 &= x^2+y^2+z^2 \\ Q_1 &= x^2+y^2+3z^2-xy \\ Q_2 &= x^2+3y^2+3z^2 \end{aligned}$$

From these forms, we can construct a partial basis for $M_{3/2}\left(36,1\right)$. Note that we have tacitly applied some inclusion-lifts; for example, $\theta^3\left(z\right)$ is also a form of level 4:

$$\begin{aligned} \theta^3\left(z\right) = \theta_{Q_0}\left(z\right) &= \sum_{n=0}^{\infty} r_{Q_0}\left(n\right)q^n \\ \theta_{Q_1}\left(z\right) &= \sum_{n=0}^{\infty} r_{Q_1}\left(n\right)q^n \\ \theta_{Q_2}\left(z\right) &= \sum_{n=0}^{\infty} r_{Q_2}\left(n\right)q^n \\ \theta^3\left(9z\right) &= \sum_{n=0}^{\infty} r_{Q_0}\left(n/9\right)q^n \end{aligned}$$

Of course, $r_{Q_0}\left(n/9\right)$ is taken to be 0 for $n \not\equiv 0 \pmod 9$. Let $\chi = \left(\frac{-3}{\cdot}\right)$. Twisting θ^3 by χ gives another form θ^3_χ in $M_{3/2}\left(36,1\right)$. Consistent with our conjecture, this form is in the space spanned by our partial basis:

$$\theta_\chi^3(z) = -1 \cdot \theta_{Q_0}(z) + 4 \cdot \theta_{Q_1}(z) - 6 \cdot \theta_{Q_2}(z) + 3\theta^3(9z)$$

Looking to coefficients, this says the following:

$$\left(\frac{-3}{n}\right) r_{Q_0}(n) = -r_{Q_0}(n) + 4r_{Q_1}(n) - 6r_{Q_2}(n) + 3r_{Q_0}(n/9)$$

This splits into three cases, depending on the value of $\left(\frac{-3}{n}\right)$. For $n \equiv 0 \pmod 3$, $\left(\frac{-3}{n}\right) = 0$. For $n \equiv 1 \pmod 3$, $\left(\frac{-3}{n}\right) = 1$. And for $n \equiv 2 \pmod 3$, $\left(\frac{-3}{n}\right) = -1$. Doing some rearranging, we have:

$$\begin{aligned} r_{Q_0}(n) &= 4r_{Q_1}(n) - 6r_{Q_2}(n) + 3r_{Q_0}(n/9) \quad \text{for} \quad n \equiv 0 \pmod 3 \\ r_{Q_0}(n) &= 2r_{Q_1}(n) - 3r_{Q_2}(n) \quad \text{for} \quad n \equiv 1 \pmod 3 \\ 2r_{Q_1}(n) &= 3r_{Q_2}(n) \quad \text{for} \quad n \equiv 2 \pmod 3 \end{aligned}$$

Further investigation has revealed some patterns. Let p be prime, and set p^* to be whichever of $\pm p$ is $1 \pmod 4$. Then $\chi_p = \left(\frac{p^*}{\cdot}\right)$ is a primitive quadratic character of conductor p. Let Q_p be the ternary quadratic form given by:

$$Q_p = x^2 + py^2 + pz^2$$

Note that Q_p has level $4p$. We have computed twists by all $p \leq 163$, and find that there are two patterns of behavior exhibited. For $p = 3, 5, 7, 13, 17, 23, 29, 41, 47, 53, 73, 83, 89, 97, 101, 107, 109, 113, 137, 149, 157$, we find that $\theta_\chi^3(z)$ is in the span of $\theta^3(z)$, $\theta^3(p^2 z)$, $\theta_{Q_p}(z)$, and possibly some other modular forms derived from ternary quadratic forms of level $4p$. In these cases, the coefficient on $\theta^3(z)$ is always -1, the coefficient on $\theta^3(p^2 z)$ is always $-p^*$, and the coefficient on $\theta_{Q_p}(z)$ is ± 6, with 6 for $p \equiv 1 \pmod 4$ and -6 for $p \equiv 3 \pmod 4$.

In the second case, for $p = 11, 19, 37, 43, 59, 61, 67, 71, 79, 103, 127, 131, 139, 151, 163$, $\theta_\chi^3(z)$ is in the span of $\theta^3(z)$, $\theta^3(p^2 z)$, and forms from quadratic forms of level $4p$, but is independent of $\theta_{Q_p}(z)$. In this case, the coefficient on $\theta^3(z)$ is always 1, and the coefficient on $\theta^3(p^2 z)$ is again $-p^*$.

As an approach to resolving the original conjecture, we would like to investigate the equivalent identities involving representation numbers. If we can make progress in this area, it may provide a path to resolving the equivalent statement about spaces of modular forms. Alternatively, if the conjecture can be resolved by other methods, a family of identities on representation numbers would follow as a corollary.

4. Appendix A - Dimension Tables

Since we have that $\dim\left(B_{3/2}(N,d)\right) \leq \dim\left(C_{3/2}(N,d)\right) \leq \dim\left(A_{3/2}(N,d)\right)$, in the cases where $\dim\left(B_{3/2}(N,d)\right) = \dim\left(A_{3/2}(N,d)\right)$, there is no need to compute $\dim\left(C_{3/2}(N,d)\right)$. For this reason, the computation of $\dim\left(C_{3/2}(N,d)\right)$ is left empty unless $\dim\left(B_{3/2}(N,d)\right) \neq \dim\left(A_{3/2}(N,d)\right)$.

Table 1. Dimensions of spaces from level 4 to 64

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
4	1	1	1	1	
8	1	2	2	2	
8	2	2	2	2	
12	1	3	3	3	
12	3	3	3	3	
16	1	4	4	4	
16	2	3	3	3	
20	1	3	3	3	
20	5	3	3	3	
24	1	5	5	5	
24	2	5	5	5	
24	3	5	5	5	
24	6	5	5	5	
28	1	4	4	4	
28	7	4	4	4	
32	1	6	6	6	
32	2	6	6	6	
36	1	6	6	6	
36	3	8	8	8	
40	1	6	6	6	
40	2	6	6	6	
40	5	6	6	6	
40	10	6	6	6	
44	1	5	5	5	
44	11	5	5	5	
48	1	10	10	10	
48	2	8	8	8	
48	3	10	10	10	
48	6	8	8	8	
52	1	5	5	5	
52	13	5	5	5	
56	1	7	7	7	
56	2	7	7	7	
56	7	7	7	7	
56	14	7	7	7	
60	1	10	10	10	
60	3	10	10	10	
60	5	10	10	10	
60	15	10	10	10	
64	1	10	10	10	
64	2	10	10	10	

Table 2. Dimensions of spaces from level 68 to 116

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
68	1	6	6	6	
68	17	6	6	6	
72	1	12	12	12	
72	2	12	12	12	
72	3	12	12	12	
72	6	12	12	12	
76	1	7	7	7	
76	19	7	7	7	
80	1	12	12	12	
80	2	10	10	10	
80	5	12	12	12	
80	10	10	10	10	
84	1	12	12	12	
84	3	12	12	12	
84	7	12	12	12	
84	21	12	12	12	
88	1	9	9	9	
88	2	9	9	9	
88	11	9	9	9	
88	22	9	9	9	
92	1	8	8	8	
92	23	8	8	8	
96	1	16	16	16	
96	2	16	16	16	
96	3	16	16	16	
96	6	16	16	16	
100	1	12	8	12	8
100	5	12	9	11	10
104	1	10	10	10	
104	2	10	10	10	
104	13	10	10	10	
104	26	10	10	10	
108	1	15	15	15	
108	3	15	15	15	
112	1	14	14	14	
112	2	12	12	12	
112	7	14	14	14	
112	14	12	12	12	
116	1	9	8	8	
116	29	9	8	8	

Table 3. Dimensions of spaces from level 120 to 160

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
120	1	18	18	18	
120	2	18	18	18	
120	3	18	18	18	
120	6	18	18	18	
120	5	18	18	18	
120	10	18	18	18	
120	15	18	18	18	
120	30	18	18	18	
124	1	10	10	10	
124	31	10	10	10	
128	1	16	16	16	
128	2	16	16	16	
132	1	16	16	16	
132	3	16	16	16	
132	11	16	16	16	
132	33	16	16	16	
136	1	12	12	12	
136	2	12	12	12	
136	17	12	12	12	
136	34	12	12	12	
140	1	16	16	16	
140	5	16	16	16	
140	7	16	16	16	
140	35	16	16	16	
144	1	24	24	24	
144	2	20	20	20	
144	3	24	24	24	
144	6	20	20	20	
148	1	11	9	9	
148	37	11	9	9	
152	1	13	13	13	
152	2	13	13	13	
152	19	13	13	13	
152	38	13	13	13	
156	1	18	18	18	
156	3	18	18	18	
156	13	18	18	18	
156	39	18	18	18	
160	1	20	20	20	
160	2	20	20	20	
160	5	20	20	20	
160	10	20	20	20	

Table 4. Dimensions of spaces from level 164 to 204

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
164	1	12	11	11	
164	41	12	11	11	
168	1	22	22	22	
168	2	22	22	22	
168	3	22	22	22	
168	6	22	22	22	
168	7	22	22	22	
168	14	22	22	22	
168	21	22	22	22	
168	42	22	22	22	
172	1	13	11	11	
172	43	13	11	11	
176	1	18	18	18	
176	2	16	15	15	
176	11	18	18	18	
176	22	16	15	15	
180	1	24	24	24	
180	3	28	28	28	
180	5	24	24	24	
180	15	28	28	28	
184	1	15	14	14	
184	2	15	14	14	
184	23	15	14	14	
184	46	15	14	14	
188	1	14	14	14	
188	47	14	14	14	
192	1	28	28	28	
192	2	28	28	28	
192	3	28	28	28	
192	6	28	28	28	
196	1	20	12	12	
196	7	24	16	16	
200	1	24	18	24	18
200	2	24	18	24	18
200	5	24	19	24	21
200	10	24	19	24	21
204	1	22	21	21	
204	3	22	21	21	
204	17	22	21	21	
204	51	22	21	21	

Table 5. Dimensions of spaces from level 208 to 248

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
208	1	20	20	20	
208	2	18	18	18	
208	13	20	20	20	
208	26	18	18	18	
212	1	15	12	12	
212	53	15	12	12	
216	1	27	27	27	
216	2	27	27	27	
216	3	27	27	27	
216	6	27	27	27	
220	1	22	22	22	
220	5	22	22	22	
220	11	22	22	22	
220	55	22	22	22	
224	1	24	24	24	
224	2	24	24	24	
224	7	24	24	24	
224	14	24	24	24	
228	1	24	22	22	
228	3	24	22	22	
228	19	24	22	22	
228	57	24	22	22	
232	1	18	16	16	
232	2	18	16	16	
232	29	18	16	16	
232	58	18	16	16	
236	1	17	16	16	
236	59	17	16	16	
240	1	36	36	36	
240	2	32	32	32	
240	3	36	36	36	
240	6	32	32	32	
240	5	36	36	36	
240	10	32	32	32	
240	15	36	36	36	
240	30	32	32	32	
244	1	17	14	14	
244	61	17	14	14	
248	1	19	18	18	
248	2	19	18	18	
248	31	19	18	18	
248	62	19	18	18	

Table 6. Dimensions of spaces from level 252 to 288

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
252	1	32	32	32	
252	3	32	32	32	
252	7	32	32	32	
252	21	32	32	32	
256	1	28	24	28	24
256	2	28	25	28	25
260	1	24	21	21	
260	5	24	21	21	
260	13	24	21	21	
260	65	24	21	21	
264	1	30	30	30	
264	2	30	30	30	
264	3	30	30	30	
264	6	30	30	30	
264	11	30	30	30	
264	22	30	30	30	
264	33	30	30	30	
264	66	30	30	30	
268	1	19	15	15	
268	67	19	15	15	
272	1	24	24	24	
272	2	22	21	21	
272	17	24	24	24	
272	34	22	21	21	
276	1	28	27	27	
276	3	28	27	27	
276	23	28	27	27	
276	69	28	27	27	
280	1	30	30	30	
280	2	30	30	30	
280	5	30	30	30	
280	10	30	30	30	
280	7	30	30	30	
280	14	30	30	30	
280	35	30	30	30	
280	70	30	30	30	
284	1	20	18	18	
284	71	20	18	18	
288	1	40	40	40	
288	2	40	40	40	
288	3	40	40	40	
288	6	40	40	40	

Table 7. Dimensions of spaces from level 292 to 328

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
292	1	20	16	16	
292	73	20	16	16	
296	1	22	18	18	
296	2	22	18	18	
296	37	22	18	18	
296	74	22	18	18	
300	1	42	34	42	34
300	3	42	34	42	34
300	5	42	35	41	38
300	15	42	35	41	38
304	1	26	26	26	
304	2	24	23	23	
304	19	26	26	26	
304	38	24	23	23	
308	1	28	25	25	
308	7	28	25	25	
308	11	28	25	25	
308	77	28	25	25	
312	1	34	33	33	
312	2	34	33	33	
312	3	34	33	33	
312	6	34	33	33	
312	13	34	33	33	
312	26	34	33	33	
312	39	34	33	33	
312	78	34	33	33	
316	1	22	18	18	
316	79	22	18	18	
320	1	36	36	36	
320	2	36	36	36	
320	5	36	36	36	
320	10	36	36	36	
324	1	36	26	26	
324	3	42	34	34	
328	1	24	22	22	
328	2	24	22	22	
328	41	24	22	22	
328	82	24	22	22	

TABLE 8. Dimensions of spaces from level 332 to 364

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
332	1	23	20	20	
332	83	23	20	20	
336	1	44	44	44	
336	2	40	40	40	
336	3	44	44	44	
336	6	40	40	40	
336	7	44	44	44	
336	14	40	40	40	
336	21	44	44	44	
336	42	40	40	40	
340	1	30	25	25	
340	5	30	25	25	
340	17	30	25	25	
340	85	30	25	25	
344	1	25	21	21	
344	2	25	21	21	
344	43	25	21	21	
344	86	25	21	21	
348	1	34	32	32	
348	3	34	32	32	
348	29	34	32	32	
348	87	34	32	32	
352	1	32	30	30	
352	2	32	30	30	
352	11	32	30	30	
352	22	32	30	30	
356	1	24	20	20	
356	89	24	20	20	
360	1	48	48	48	
360	2	48	48	48	
360	3	48	48	48	
360	6	48	48	48	
360	5	48	48	48	
360	10	48	48	48	
360	15	48	48	48	
360	30	48	48	48	
364	1	32	28	28	
364	7	32	28	28	
364	13	32	28	28	
364	91	32	28	28	

Table 9. Dimensions of spaces from level 368 to 404

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
368	1	30	28	28	
368	2	28	24	24	
368	23	30	28	28	
368	46	28	24	24	
372	1	36	32	32	
372	3	36	32	32	
372	31	36	32	32	
372	93	36	32	32	
376	1	27	25	25	
376	2	27	25	25	
376	47	27	25	25	
376	94	27	25	25	
380	1	34	32	32	
380	5	34	32	32	
380	19	34	32	32	
380	95	34	32	32	
384	1	48	48	48	
384	2	48	48	48	
384	3	48	48	48	
384	6	48	48	48	
388	1	26	20	20	
388	97	26	20	20	
392	1	40	28	28	
392	2	40	28	28	
392	7	40	28	28	
392	14	40	28	28	
396	1	44	40	40	
396	3	44	43	43	
396	11	44	40	40	
396	33	44	43	43	
400	1	48	36	48	36
400	2	42	34	42	34
400	5	48	38	48	42
400	10	42	35	42	38
404	1	27	22	22	
404	101	27	22	22	

Table 10. Dimensions of spaces from level 408 to 440

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
408	1	42	40	40	
408	2	42	40	40	
408	3	42	40	40	
408	6	42	40	40	
408	17	42	40	40	
408	34	42	40	40	
408	51	42	40	40	
408	102	42	40	40	
412	1	28	24	24	
412	103	28	24	24	
416	1	36	36	36	
416	2	36	36	36	
416	13	36	36	36	
416	26	36	36	36	
420	1	56	55	55	
420	3	56	55	55	
420	5	56	55	55	
420	15	56	55	55	
420	7	56	55	55	
420	21	56	55	55	
420	35	56	55	55	
420	105	56	55	55	
424	1	30	24	24	
424	2	30	24	24	
424	53	30	24	24	
424	106	30	24	24	
428	1	29	22	22	
428	107	29	22	22	
432	1	54	54	54	
432	2	48	47	47	
432	3	54	54	54	
432	6	48	47	47	
436	1	29	20	20	
436	109	29	20	20	
440	1	42	41	41	
440	2	42	41	41	
440	5	42	41	41	
440	10	42	41	41	
440	11	42	41	41	
440	22	42	41	41	
440	55	42	41	41	
440	110	42	41	41	

Table 11. Dimensions of spaces from level 444 to 476

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
444	1	42	38	38	
444	3	42	38	38	
444	37	42	38	38	
444	111	42	38	38	
448	1	44	41	41	
448	2	44	44	44	
448	7	44	41	41	
448	14	44	44	44	
452	1	30	21	21	
452	113	30	21	21	
456	1	46	42	42	
456	2	46	42	42	
456	3	46	42	42	
456	6	46	42	42	
456	19	46	42	42	
456	38	46	42	42	
456	57	46	42	42	
456	114	46	42	42	
460	1	40	36	36	
460	5	40	36	36	
460	23	40	36	36	
460	115	40	36	36	
464	1	36	32	32	
464	2	34	28	28	
464	29	36	32	32	
464	58	34	28	28	
468	1	48	46	46	
468	3	52	50	50	
468	13	48	46	46	
468	39	52	50	50	
472	1	33	30	30	
472	2	33	30	30	
472	59	33	30	30	
472	118	33	30	30	
476	1	40	38	38	
476	7	40	38	38	
476	17	40	38	38	
476	119	40	38	38	

TABLE 12. Dimensions of spaces from level 480 to 516

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
480	1	64	64	64	
480	2	64	64	64	
480	3	64	64	64	
480	6	64	64	64	
480	5	64	64	64	
480	10	64	64	64	
480	15	64	64	64	
480	30	64	64	64	
484	1	42	18	18	
484	11	48	30	30	
488	1	34	28	28	
488	2	34	28	28	
488	61	34	28	28	
488	122	34	28	28	
492	1	46	39	39	
492	3	46	39	39	
492	41	46	39	39	
492	123	46	39	39	
496	1	38	36	36	
496	2	36	32	32	
496	31	38	36	36	
496	62	36	32	32	
500	1	45	32	42	35
500	5	45	32	43	35
504	1	60	60	60	
504	2	60	60	60	
504	3	60	59	59	
504	6	60	59	59	
504	7	60	60	60	
504	14	60	60	60	
504	21	60	59	59	
504	42	60	59	59	
508	1	34	26	26	
508	127	34	26	26	
512	1	48	39	48	39
512	2	48	39	48	39
516	1	48	40	40	
516	3	48	40	40	
516	43	48	40	40	
516	129	48	40	40	

Table 13. Dimensions of spaces from level 520 to 552

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
520	1	48	43	43	
520	2	48	43	43	
520	5	48	43	43	
520	10	48	43	43	
520	13	48	43	43	
520	26	48	43	43	
520	65	48	43	43	
520	130	48	43	43	
524	1	35	29	29	
524	131	35	29	29	
528	1	60	60	60	
528	2	56	54	54	
528	3	60	60	60	
528	6	56	54	54	
528	11	60	60	60	
528	22	56	54	54	
528	33	60	60	60	
528	66	56	54	54	
532	1	44	36	36	
532	7	44	36	36	
532	19	44	36	36	
532	133	44	36	36	
536	1	37	29	29	
536	2	37	29	29	
536	67	37	29	29	
536	134	37	29	29	
540	1	66	64	64	
540	3	66	64	64	
540	5	66	64	64	
540	15	66	64	64	
544	1	44	42	42	
544	2	44	42	42	
544	17	44	42	42	
544	34	44	42	42	
548	1	36	25	25	
548	137	36	25	25	
552	1	54	50	50	
552	2	54	50	50	
552	3	54	50	50	
552	6	54	50	50	
552	23	54	50	50	
552	46	54	50	50	
552	69	54	50	50	
552	138	54	50	50	

TABLE 14. Dimensions of spaces from level 556 to 576

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
556	1	37	28	28	
556	139	37	28	28	
560	1	60	60	60	
560	2	56	54	54	
560	5	60	60	60	
560	10	56	54	54	
560	7	60	60	60	
560	14	56	54	54	
560	35	60	60	60	
560	70	56	54	54	
564	1	52	45	45	
564	3	52	45	45	
564	47	52	45	45	
564	141	52	45	45	
568	1	39	32	32	
568	2	39	32	32	
568	71	39	32	32	
568	142	39	32	32	
572	1	46	42	42	
572	11	46	42	42	
572	13	46	42	42	
572	143	46	42	42	
576	1	72	72	72	
576	2	72	72	72	
576	3	73	73	73	
576	6	72	72	72	

Table 15. Dimensions of spaces from level 580 to 600

Level	Character	$M_{3/2}(N,d)$	$B_{3/2}(N,d)$	$A_{3/2}(N,d)$	$C_{3/2}(N,d)$
580	1	48	39	39	
580	5	48	39	39	
580	29	48	39	39	
580	145	48	39	39	
584	1	40	32	32	
584	2	40	32	32	
584	73	40	32	32	
584	146	40	32	32	
588	1	72	48	48	
588	3	72	48	48	
588	7	72	54	54	
588	21	72	54	54	
592	1	44	36	36	
592	2	42	34	34	
592	37	44	36	36	
592	74	42	34	34	
596	1	39	28	28	
596	149	39	28	28	
600	1	78	66	78	66
600	2	78	66	78	66
600	3	78	66	78	66
600	6	78	66	78	66
600	5	78	67	78	72
600	10	78	67	78	72
600	15	78	67	78	72
600	30	78	67	78	72

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