CLASSIFYING EXTENSIONS OF THE FIELD OF FORMAL LAURENT SERIES OVER $\mathbb{F}_p$

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Abstract. In previous works, Jones-Roberts and Pauli-Roblot have studied finite extensions of the p-adic numbers $\mathbb{Q}_p$. This paper focuses on results for local fields of characteristic p. In particular we are able to produce analogous results to Jones-Roberts in the case that the characteristic does not divide the degree of the field extension. Also in this case, following from the work of Pauli-Roblot, we prove that the defining polynomials of these extensions can be written in a simple form amenable to computation. Finally, if p is the degree of the extension, we show there are infinitely many extensions of this degree and thus these cannot be classified in the same manner.

1. Introduction

Classifying extensions of $\mathbb{Q}_p$ has been of interest for many years. Pauli and Roblot [11] describe a method for computing defining polynomials for all extensions of $\mathbb{Q}_p$ of a given degree. Jones and Roberts [9] constructed an online database that identifies degree n extensions of $\mathbb{Q}_p$ for small values of p and n. They describe how to compute various invariants for each extension, including the Galois group.

In a similar fashion, we extend these results to characteristic p local fields, focusing on the unramified, totally tamely ramified, and totally wildly ramified cases. We begin by introducing the reader to essential background topics.

Given a characteristic p local field F and an integer n relatively prime to p we classify all degree n extensions of F. We recall the result that for each $f \mid n$ there is a unique unramified extension K of degree f. We then turn our attention to totally tamely ramified extensions of K degree $e = n/f$. We follow the work of Jones and Roberts [9] to compute a class of defining polynomials for these extensions, namely a specific type of Eisenstein polynomial.

We next consider the totally wildly ramified case when $n = p$. Our results for degree p extensions are not analogous to the case of characteristic zero local fields, as there are infinitely many degree p extensions.

We conclude by classifying all degree 10 extensions of $\mathbb{F}_p((T))$ where $p \equiv \pm 3 \pmod{10}$. In particular, in the case that $p = 3$ we give specific defining polynomials for each extension. This illustrates computationally how one handles a specific degree and characteristic.

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2. Background

2.1. **Local Fields.** This paper will be concerned with extensions of local fields. We refer the reader to [6, 14] for more details on local fields.

Let F be a local field. Let π_F denote a uniformizer of F and write $\mathcal{O}_F$ for the valuation ring of F, $\mathcal{M}_F = (\pi_F)$ for the maximal ideal, and residue field $\mathcal{O}_F/\mathcal{M}_F$. We normalize the valuation on F so that $\nu_F(\pi_F) = 1$.

Throughout this paper L/F will always refer to a finite extension of local fields. Given L/F of degree n one has $\pi_F = \pi_L^e$ for some integer $e \geq 1$ with $e \mid n$. We call e the *ramification index* of L/F and $f = n/e$ the *inertia degree* of L/F. We say L/F is *unramified* if $e = 1$ and *totally ramified* if $e = n$. If $\mathcal{O}_F/\mathcal{M}_F$ has characteristic p, we say L/F is *tamely ramified* if $p \nmid e$ and *wildly ramified* if $p \mid e$.

One knows that the compositum of unramified extensions is again unramified, so one can form a maximal unramified extension F^{ur} of F. Given an extension L/F of local fields, we set $K = L \cap F^{\mathrm{ur}}$. Clearly K is the maximal unramified extension of F in L. Note the extension L/K is necessarily totally ramified.

2.2. **The Field of Formal Laurent Series.** We will be interested in finite extensions of the field of formal Laurent series. We now introduce this field.

Let $\mathbb{F}_p[T]$ be the polynomial ring with coefficients in $\mathbb{F}_p$ and $\mathbb{F}_p(T)$ its fraction field.

Definition 2.1. Given $x \in \mathbb{F}_p(T)$, write x as $T^r \frac{g}{h}$ with $g, h \in \mathbb{F}_p[T]$, $T \nmid gh$. We define a valuation ν_T by:

$$\nu_T\left(T^r \frac{g}{h}\right) = r$$

with $\nu_T(0) = \infty$.

Note that we can define ν_f for any irreducible polynomial in $\mathbb{F}_p[T]$ analogously. As the valuations arising in this manner are non-Archimedean, they give the characteristic p valuations analogous to the p-adic valuations on $\mathbb{Q}$. Moreover, one can define a valuation with respect to $1/T$ to obtain the characteristic p valuation analogous to the usual absolute value on $\mathbb{Q}$. As we will only be interested in the case $f = T$, we restrict to that case.

We can now complete $\mathbb{F}_p(T)$ with respect to ν_T to obtain the field of formal Laurent series over $\mathbb{F}_p$.

Definition 2.2. A *formal Laurent series* $f(T)$ is an infinite series of the form

$$\sum_{i=-m}^{\infty} a_i T^i$$

with $m, i \in \mathbb{Z}$, $a_i \in \mathbb{F}_p$ for all i. We denote the set of such series by $\mathbb{F}_p((T))$.

An equivalent expression for the valuation defined above is

$$\nu_T(x) = \nu_T\left(\sum_{i=-m}^{\infty} a_i T^i\right) = -m.$$

We also define an absolute value $|\cdot|_T$ such that $\left|T^r \frac{g}{h}\right|_T = p^{-r}$.

Note that $\mathbb{F}_p((T))$ is a non-Archimedean local field with characteristic p. As we will only discuss the valuation on $\mathbb{F}_p((T))$, we will be using the notation $\nu(x)$ rather than $\nu_T(x)$ to denote this specific valuation for the remainder of the paper unless otherwise specified. Given an extension $L/\mathbb{F}_p((T))$ we denote the valuation on L obtained by extending ν by ν_L.

For the rest of the paper all our fields will be extensions of $\mathbb{F}_p((T))$ for some prime p. In particular, F will be fixed to be a finite extension of $\mathbb{F}_p((T))$.

2.3. **Ramification Groups.** Let L/F be a Galois extension of local fields with Galois group G. We define the ramification groups of L/F by

$$G_i = \{\sigma \in G : \nu_L(\sigma(x) - x) \geq i + 1 \text{ for all } x \in \mathcal{O}_L\}$$

where $i \geq -1$. The ramification groups make up a chain of subgroups of the Galois group that are eventually trivial. These G_i may not be distinct for all i.

Definition 2.3. In the subgroup chain of ramification groups, a *ramification break* is defined to occur at $i \geq 0$ such that $G_i \neq G_{i+1}$.

Depending on the Galois group and ramification groups themselves, this break may be unique. Note that the chain of ramification groups is an invariant of the field, so distinct chains give distinct fields.

3. Unramified Extensions

Unramified extensions of characteristic p fields are similar to their characteristic zero counterparts. We have the following theorem in this regard.

Theorem 3.1. [8, p. 167] Let F be a local field and f be a positive integer. Then F has a unique unramified extension of degree f. This extension is obtained by adjoining a primitive $(p^f - 1)$st root of unity to F.

In particular, we see that if we wish to classify extensions of degree n of a local field F, it is enough to classify all the totally ramified extensions of degree e for each $e \mid n$.

4. Totally Ramified Extensions

As noted in the previous section, unramified extensions are already well understood. Thus when we build up our degree n extension of F we focus on building totally ramified extensions of degree e for each $e \mid n$.

Definition 4.1. Let $g(x) \in \mathcal{O}_F[x]$ be a monic polynomial:

$$g(x) = x^e + a_{e-1}x^{e-1} + ... + a_0.$$

If $\nu(a_i) \geq 1$ for each $i = 0, ..., e-1$, and $\nu(a_0) = 1$, then $g(x)$ is said to be *Eisenstein*.

The following is a well-known theorem which describes how to construct totally ramified extensions.

Theorem 4.2. [6, p. 54] A finite extension L/K of a non-Archimedean local field is totally ramified if and only if $L = K[\alpha]$, with α a root of an Eisenstein polynomial.

4.1. **Totally Tamely Ramified Extensions.** Using the work of Pauli and Roblot [11], we can show exactly what the totally tamely ramified extensions look like, but first we need some theorems adapted from Pauli [12]. We let K/F be an unramified extension of degree f and consider totally tamely ramified extensions L/K of degree e. We define $|\mathcal{M}_K|_K := |\pi_K|_K$.

Definition 4.3. Let L/K be a degree e Galois extension with Galois group G. Let $(\delta_1, \cdots, \delta_e)$ be an integral basis of L/K. Write $G = \{\sigma_1, \ldots, \sigma_e\}$. Then

$$\operatorname{disc}(L/K) = (\det(\sigma_l(\delta_k))_{1\le k\le e, 1\le l\le e})^2$$

is the *discriminant* of L/K.

The discriminant of the field generated by an Eisenstein polynomial is exactly the discriminant of the polynomial.

Lemma 4.4. Let $L = K(\alpha)/K$ be a finite Galois extension of degree e with basis elements $1, x, x^2, \cdots, x^{e-1}$ and g be the minimal polynomial over K with roots $\alpha_1, \ldots, \alpha_e$ where $\alpha = \alpha_1$. Then $\operatorname{disc}(L/K) = \operatorname{disc}(g)$ and $\nu_K(\operatorname{disc}(g)) = e\nu_K(g'(\alpha))$.

Proof. Define $\sigma_i \in \operatorname{Gal}(L/K)$ such that $\sigma_i(\alpha) = \alpha_i$ for $i \in \{1, \ldots, e\}$. Then $\sigma_i(x^j) = \alpha_i^j$ for $0 \le j \le e-1$. Note $\operatorname{disc}(L/K)$ is the square of the determinant of the matrix

$$A = \begin{bmatrix} 1 & x_1 & \ldots & x_1^{e-1} \\ 1 & x_2 & \ldots & x_2^{e-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_e & \ldots & x_e^{e-1} \end{bmatrix}.$$

Since A is a Vandermonde matrix, $\det A = \prod_{i<j}(\alpha_i - \alpha_j)$ and it follows that $\operatorname{disc}(L/K) = \operatorname{disc}(g)$. On the other hand, for any $y \in L$ we can write $g(y) = (y-\alpha_1)\cdots(y-\alpha_n)$, so we have

$$g'(\alpha_i) = \sum_k \prod_{j\ne k}(\alpha_i - \alpha_j).$$

However, only the $k = i$ term is non-zero. Hence

$$g'(\alpha_i) = \prod_{j\ne i}(\alpha_i - \alpha_j),$$

so it follows that

$$\operatorname{disc}(g) = \prod_{i=1}^{e} g'(\alpha_i).$$

Therefore,

$$\nu_K(\operatorname{disc}(g)) = \nu_K(\prod_{i=1}^{e} g'(\alpha_i)) = e\nu_K(g'(\alpha_i)).$$

$\square$

Lemma 4.5. If $x_0, \cdots, x_{e-1} \in K$ where $|x_i|_K \ne |x_j|_K$ for $i \ne j$, then

$$\left|\sum_{i=0}^{e-1} x_i\right|_K = \max_{0\le i\le e-1}\{|x_i|_K\}.$$

Theorem 4.6. (Ore's Conditions) For each $e \mid n$ there exists a totally ramified extension L/K of degree e and discriminant $\mathcal{M}_K^{e-1}$.

Proof. By Theorem 4.2, every totally ramified extension L of K of degree e can be generated by adjoining a root α of an Eisenstein polynomial $g(x) = x^e + a_{e-1}x^{e-1} + ... + a_0$. We have $\mathrm{disc}(L/K) = \mathrm{disc}(g(x))$ and since $g(x)$ is Eisenstein, we can write $\nu_K(\mathrm{disc}(g(x)))/e = \nu_K(g'(\alpha))$ because $g(x)$ is irreducible. Since α is a uniformizer in L, $\nu_K(\alpha) = 1/e$. The valuations of $ia_i\alpha^{i-1}$ for $1 \le i < e$ and $e\alpha^{e-1}$ are all different and so by Lemma 4.5 we get

$$\begin{aligned}\nu_K(g'(\alpha)) &= \nu_K(e\alpha^{e-1} + (e-1)a_{e-1}\alpha^{e-1} + ... + a_1) \\ &= \min_{1\le i\le e-1}\left\{\nu_K(e) + \frac{e-1}{e}, \nu_K(i) + \nu_K(a_i) + \frac{i-1}{e}\right\}.\end{aligned}$$

Note that $\nu_K(x) = 0$ for all $x \in \mathbb{Z}$ and $\nu_K(a_i) \ge 1$ for all $1 \le i \le e-1$, so

$$\begin{aligned}\nu_K(g'(\alpha)) &= \min_{1\le i\le e-1}\left\{\frac{e-1}{e}, \nu_K(a_i) + \frac{i-1}{e}\right\} \\ &= \frac{e-1}{e}.\end{aligned}$$

Thus since $g(x)$ is irreducible and $\nu_K(\mathrm{disc}(g(x))) = e\nu_K(g'(\alpha)) = e-1$ it is clear that we can construct an Eisenstein polynomial $g(x)$ such that $\mathrm{disc}(g(x)) = \mathcal{M}_K^{e-1}$. □

4.2. **Construction of Generating Polynomials.** Let $\mathbf{L}_e$ denote the set of all totally ramified extensions L/K of degree e and discriminant $\mathcal{M}_K^{e-1}$. In this section we use the work of [11,12] to construct a finite set of polynomials that will generate all the extensions in $\mathbf{L}_e$. As above, we let K/F be an unramified extension of degree f and L/K be a totally ramified extension of degree e. Let H be the Galois group of the extension K/F and let $\mathcal{R}_{1,2}$ be a fixed H-stable system of representatives of the quotient $\mathcal{M}_K^1/\mathcal{M}_K^2$. We denote $\mathcal{R}_{1,2}^*$ to be the subset of $\mathcal{R}_{1,2}$ whose elements have ν_K-valuation 1.

Let Ω be the set of e-tuples $(\omega_0, \dots, \omega_{e-1}) \in K^e$ which satisfy the following conditions:

$$\omega_i \in \begin{cases} \mathcal{R}_{1,2}^* & \text{if } i = 0, \\ \mathcal{R}_{1,2} & \text{if } 1 \le i \le e-1. \end{cases} \tag{1}$$

To each element $\omega = (\omega_0, \dots, \omega_{e-1}) \in \Omega$ we associate the polynomial $A_\omega(x) \in \mathcal{O}_K[x]$ given by

$$A_\omega(x) = x^e + \omega_{e-1}x^{e-1} + \cdots + \omega_1 x + \omega_0.$$

Lemma 4.7. The polynomials A_ω are Eisenstein polynomials of discriminant $\mathcal{M}_K^{e-1}$.

Proof. By construction $\nu_K(\omega_i) \ge 1$ for all i and $\nu_K(\omega_0) = 1$. So A_ω is an Eisenstein polynomial.

Let α be a root of A_ω. Since the discriminant of A_ω is the norm from $K(\alpha)$ to K of $A'_w(\alpha)$ we have

$$\nu_K(A'_w(\alpha)) = \frac{e-1}{e}$$

as seen in Theorem 4.6. It follows that $\nu_K(\mathrm{disc}(A_\omega)) = e-1$ and $\mathrm{disc}(A_\omega) = \mathcal{M}_K^{e-1}$ as claimed. □

Lemma 4.8. Let ω be an element of Ω and let α be a root of $A_\omega(x)$. The extension $K(\alpha)/K$ is a totally ramified extension of degree e and discriminant $\mathcal{M}_K^{e-1}$. Conversely, if L/K is totally ramified extension of degree e and discriminant $\mathcal{M}_K^{e-1}$, then there exists $\omega \in \Omega$ and a root α of $A_\omega(x)$ such that $L = K(\alpha)$.

Proof. The statement is a special case of the characteristic zero result Corollary 5.3 in [11]. In particular, one specializes to $j = 0$ and $c = 2$. The proof there works for characteristic p as well. □

Theorem 4.9. Let q be the order of the residue field of K. Then the number of totally ramified extensions of K of degree e and discriminant $\mathcal{M}_K^{e-1}$ is

$$\#\mathbf{L}_e = e.$$

Proof. To see this, one combines Lemma 6.2 and Lemma 6.3 of [11] and observes the proofs carry over verbatim to characteristic p. □

Pauli and Roblot have calculated convenient polynomials that generate totally tamely ramified extensions of unramified extensions of $\mathbb{Q}_p$. Their proof carries over to the positive characteristic case as well. We include the proof for the convenience of the reader.

Theorem 4.10. Let ζ be a primitive $(p^f - 1)$-st root of unity contained in K and let $g = \gcd(p^f - 1, e)$. Set $m = e/g$. There are exactly e totally and tamely ramified extensions of K of degree e. Furthermore, these extensions can be split into g classes of m K-isomorphic extensions, all extensions in the same class being generated over K by the roots of the polynomials

$$f_r(x) = x^e - \zeta^r \pi_K$$

for $r = 0, ..., g - 1$.

Proof. Consider the set $\mathcal{R}_{1,2}^* = \{\zeta^i \pi_K \text{ with } 0 \le i \le p^f - 2\}$ and $\mathcal{R}_{1,2} = \mathcal{R}_{1,2}^* \cup \{0\}$. The roots of the polynomials $x^e + \omega_{e-1}x^{e-1} + ... + \omega_0$, where $\omega_i \in \mathcal{R}_{1,2}$ for $1 \le i \le e-1$ and $\omega_0 \in \mathcal{R}_{1,2}^*$, generate all totally tamely ramified extensions of discriminant $\mathcal{M}_K^{e-1}$ by Lemma 4.8.

Consider extensions of K generated by roots of the polynomials $f_i(x) = x^e - \zeta^i \pi_K$ so that $\omega_j = 0$ for $1 \le j \le e - 1$. Let α be a root of $f_i(x)$. Note that since $\zeta \in K$, we have $\zeta^h \alpha$ generates the same extension of K as α for any integer h. If we choose h so that $eh + i \equiv r \pmod{p^f - 1}$ with $0 \le r < g$, then the minimal polynomial of $\zeta^h \alpha$ is $f_{eh+i}(x)$ since

$$\begin{aligned} (\zeta^h \alpha)^e + \zeta^{eh+i}\pi_K &= \zeta^{eh}\alpha^e + \zeta^{eh+i}\pi_K \\ &= \zeta^{eh}(\alpha^e + \zeta^i \pi_K). \end{aligned}$$

Hence we only need to consider the polynomials $f_r(x)$ for $0 \le r \le g-1$. This polynomial is Eisenstein and by Theorem 4.2, it will define a totally tamely ramified extension.

Let $f_r(x)$ and $f_{r'}(x)$ be two of these polynomials which generate a totally tamely ramified extension where $0 \le r, r' \le g - 1$ and $r \ne r'$. Let α and α' be roots of $f_r(x)$ and $f_{r'}(x)$ respectively. Suppose that α and α' generate the same field L. Then this field contains an e-th root of $\zeta^{r-r'}$. To see this, consider the following: If we assume $\alpha \in L$ if and only if

$\alpha' \in L$ then $f_r(\alpha) = 0 = f_{r'}(\alpha')$. Thus

$$\begin{aligned} \alpha^e - (\alpha')^e &= \zeta^{r'}\pi_K - \zeta^r \pi_K \\ &= \pi_K(\zeta^{r'} - \zeta^r) \\ &= \zeta^{r'}\pi_K(1 - \zeta^{r-r'}). \end{aligned}$$

Thus this field contains an e-th root of $\zeta^{r-r'}$ which contradicts our assumption that the field only contains the $(p^f - 1)$-st roots of unity as $r - r'$ is never a multiple of e modulo $p^f - 1$. Therefore α and α' must generate two distinct extensions of K.

Let ρ be a primitive e-th root of unity in the algebraic closure of $\mathbb{F}_p((T))$ such that for m, $\rho^m = \zeta^{(p^f-1)/g}$. The conjugates of α over K are $\alpha, \rho\alpha, ..., \rho^{e-1}\alpha$. Thus $\alpha, \rho^m\alpha, ..., \rho^{(g-1)m}\alpha$ all generate the same field, but $\alpha, \rho\alpha, ...\rho^{m-1}\alpha$ all generate distinct isomorphic extensions. More specifically, the roots of the polynomial $f_r(x)$ generate g classes of m distinct isomorphic extensions. Thus there are e total extensions generated by the roots of these polynomials. By Theorem 4.9 there are exactly e totally ramified extensions of degree e of K, which proves that all totally tamely ramified extensions of degree e of K are generated by the roots of the polynomials $f_r(x)$ as claimed. □

Thus, we have shown that the polynomials calculated in [11] to generate totally tamely ramified extensions of K of degree e where $p \nmid e$ also work in the case of $\mathrm{char}(K) = p$.

4.3. **Totally Wildly Ramified Extensions of Degree p.** In this section we discuss wildly ramified extensions L/F of degree p. We show that the characteristic p theory differs significantly from characteristic zero theory and thus it is not possible to classify such extensions as in some characteristic zero cases [2–4]. Artin-Schreier theory provides the results needed for these extensions. From this theory, the Galois group $G = \mathrm{Gal}(L/F)$ will be cyclic, namely $\mathbb{Z}/p\mathbb{Z}$. Because of that fact, the ramification groups will either be G or $\{1\}$ causing there to be a single, unique ramification break. For more on Artin-Schreier theory, see [6, p. 67-78].

Note also that in this section, the group $\mathcal{U}_i$, which corresponds to the ramification group G_i, will be written as either $1 + (\pi_L^i)$ or $1 + \mathcal{M}_L^i$.

Definition 4.11. For F a field of characteristic p, an *Artin-Schreier polynomial* is a polynomial of the form $\wp(x) = x^p - x - \alpha$ for $\alpha \in F^\times$.

The following is a well-known result that leads to our next theorem.

Lemma 4.12 (Hilbert's Theorem 90, Additive Form)**.** Let L/F be a cyclic Galois extension with degree n and Galois group G. Let σ be a generator of G and let $\beta \in L$. Then $\mathrm{Tr}_{L/F}(\beta)$ is equal to 0 if and only if there exists $\alpha \in F$ such that $\beta = \alpha - \sigma(\alpha)$.

Theorem 4.13. [10, p. 290] Any Galois extension of F of degree p is the splitting field of an Artin-Schreier polynomial.

Proof. Let L/F be a Galois extension of degree p. Then $\mathrm{Tr}_{L/F}(-1) = p(-1) = 0$ since F has characteristic p. Let σ be a generator of G. By Hilbert's Theorem 90 there exists $\alpha \in L$ such that $\sigma(\alpha) - \alpha = 1$. Thus $\sigma(\alpha) = \alpha + 1$ and $\sigma^i(\alpha) = \alpha + i$ for $i = 1, \ldots, p$. Since α has p distinct conjugates, $[F(\alpha) : F] \geq p$. It follows that $L = F(\alpha)$. Note that

$$\sigma(\alpha^p - \alpha) = \sigma(\alpha)^p - \sigma(\alpha) = (\alpha + 1)^p - (\alpha + 1) = \alpha^p - \alpha.$$

Since $\alpha^p - \alpha$ is fixed by σ, the generator of G, it is fixed by every element of G. Hence $\alpha^p - \alpha \in F$. Let $a = \alpha^p - \alpha$. Then α satisfies the equation $x^p - x - a = 0$ and L/F is the splitting field of an Artin-Schreier polynomial. $\square$

Theorem 4.14. There are infinitely many wildly ramified extensions of degree p of F.

Proof. Let L be the splitting field of the polynomial $\wp(x) = x^p - x - \pi_F^{-m} \in F[x]$ with $m \in \mathbb{Z}$. Suppose L/F is a wildly ramified extension of degree p with ν_L a discrete valuation on L and G the Galois group. Let $\pi_L \in L$ be a uniformizer. It suffices to show that there are an infinite number of values at which the unique ramification break can occur.

Consider $\nu_L(\sigma(\pi_L) - \pi_L) = 1 + \nu_L\left(\frac{\sigma(\pi_L)}{\pi_L} - 1\right)$. With this equality, in G_i we can look at $\nu_L\left(\frac{\sigma(x)}{x} - 1\right) \geq i$ rather than $\nu_L(\sigma(x) - x) \geq i + 1$. It can be found in [14, p. 67] that $\frac{\sigma(\pi_L)}{\pi_L} \in \mathcal{U}_L$. Thus, $\frac{\sigma(\pi_L)}{\pi_L} = u$ for some unit $u \in \mathcal{U}_L$. Let $u = u_F w$ for $u_F \in \mathcal{U}_F$ and $w \in 1 + \mathcal{M}_L$. Then we have,

$$\begin{aligned}\sigma\left(\frac{\sigma(\pi_L)}{\pi_L}\right) \cdot \frac{\sigma(\pi_L)}{\pi_L} &= \sigma(u_F w) \cdot u_F w \\ &= u_F^2 w \cdot \sigma(w).\end{aligned}$$

Continue this process of multiplying by $\frac{\sigma(\pi_L)}{\pi_L} = \sigma(u_F w)$ on each side until, on the left hand side, the term is equal to $\frac{\sigma^p(\pi_L)}{\pi_L}$. Because this is a degree p extension with cyclic Galois group,

$$1 = \frac{\sigma^p(\pi_L)}{\pi_L} = u_F^p w\sigma(w) \cdots \sigma^{p-1}(w) \text{ where } w\sigma(w) \cdots \sigma^{p-1}(w) \in 1 + \mathcal{M}_L.$$

Divide by $w\sigma(w) \cdots \sigma^{p-1}(w)$ to see $u_F^p \in 1 + \mathcal{M}_L$. This implies $u_F \in 1 + \mathcal{M}_L$ and $u_F \in 1 + \mathcal{M}_F$. Then, $\frac{\sigma(\pi_L)}{\pi_L} \in 1 + \mathcal{M}_L$. This gives $\frac{\sigma(\pi_L)}{\pi_L} = 1 + u_L \pi_L^s$ for some $u_L \in \mathcal{U}_L$ and $s \geq 1$, where s does not depend of choice of uniformizer. From [14, p. 66-67], $\frac{\sigma(u)}{u} \equiv 1 \pmod{\pi_L^{s+1}}$ for $u \in \mathcal{U}_L$. We can conclude for any $\lambda \in L^\times$, $\frac{\sigma(\lambda)}{\lambda} \in 1 + \pi_L^s \mathcal{U}_L$. To see this let $\lambda = u_L \pi_L^a$ with $p \nmid a$. Then

$$\begin{aligned}\frac{\sigma(\lambda)}{\lambda} &= \frac{\sigma(u_L \pi_L^a)}{u_L \pi_L^a} \\ &= \frac{\sigma(u_L)}{u_L}\left(\frac{\sigma(\pi_L)}{\pi_L}\right)^a \in 1 + \pi_L^s \mathcal{U}_L.\end{aligned}$$

Thus, $\nu_L\left(\frac{\sigma(\lambda)}{\lambda} - 1\right) = s$. This implies that $G = G_s$ and $G_{s+1} = \{1\}$. Therefore, the unique ramification break occurs at $i = s$.

Now suppose λ is a root of $\wp(x) = x^p - x - \alpha$, where $\alpha = \pi_F^{-m}$. Then,

$$\alpha = \lambda(\lambda + 1) \cdots (\lambda + (p - 1))$$

because if λ is a root, then $\lambda + j$ for $j \in \mathbb{Z}/p\mathbb{Z}$ is a root. In the above product, $(\lambda + 1), \cdots, (\lambda + (p-1))$ are units, so $\nu_F(\alpha) = \nu_L(\lambda)$. Therefore, $\nu_F(\alpha) = s$. For $\alpha = \pi_F^{-m}$, $-m = s$. Because there are infinitely many choices for m, there are infinitely many possible ramification breaks, thus extensions of degree p. $\square$

Note that when given two Artin-Schreier polynomials $\wp_1(x) = x^p - x - a$ and $\wp_2(x) = x^p - x - b$ for $a,\ b \in F$, $\nu(a) = \nu(b)$ does not imply the extensions generated by $\wp_1$ and $\wp_2$ are isomorphic. If the constant terms a and b differ by a function of the form $c^p - c$ for $c \in F$, then $\wp_1$ and $\wp_2$ will generate isomorphic extensions.

5. Example

We utilize the results proven in the paper to classify all degree $n = 10$ field extensions L/F where $F = \mathbb{F}_p((T))$ with $p \equiv \pm 3 \pmod{10}$. We have L/F is necessarily one of the following:

(1) a degree 10 unramified extension,
(2) a degree 2 totally tamely ramified extension of a degree 5 unramified extension,
(3) a degree 5 totally tamely ramified extension of a degree 2 unramified extension,
(4) or a degree 10 totally tamely ramified extension.

From Theorem 3.1, the unramified portion of each case is unique. These extensions are formed by adjoining a root of the cyclotomic polynomial $x^{p^f} - x$ and have Galois group isomorphic to $\mathbb{Z}/f\mathbb{Z}$. To compute a defining polynomial for these extensions, see [5, p. 587] which uses an algorithm to find irreducible polynomials in the ring $\mathbb{F}_p[x]$ that can be applied to the polynomial ring over $\mathbb{F}_p((T))$.

For the totally tamely ramified portion of the extensions, it is necessary to use a formula similar to the one for the characteristic zero case outlined in [11]. By Theorem 4.9 there are e distinct, but not necessarily non-isomorphic degree e extensions. By Theorem 4.10 for $g = \gcd(e, p^f - 1)$ there are g non-isomorphic totally tamely ramified extensions of degree e and the defining polynomials are in the form $x^e - \zeta^r \pi_F$ for $0 \le r \le g - 1$. Thus for case 1 there is 1 unique extension and there are $\gcd(2, p^5 - 1) = 2$, $\gcd(5, p^2 - 1) = 1$, $\gcd(10, p^1 - 1) = 2$, non-isomorphic extensions for case 2, 3 and 4 respectively. In total, there are 6 non-isomorphic extensions of degree 10 for such p.

To calculate the Galois group of each of these extensions, it is necessary to use a lemma found in [14, p. 66-67]:

Lemma 5.1. Let F be a field of characteristic p. Let L/F be a Galois extension with Galois group G and let $\mathcal{M}_L$ denote the maximal ideal of the integers in L. For $i \ge -1$, let G_i be the i-th ramification group. Let U_0 be the units in L and for $i \ge 1$, let $U_i = 1 + (\pi_L^i)$, where π_L is the generator of $\mathcal{M}_L$.

(a) For $i \ge 0$, G_i/G_{i+1} is isomorphic to a subgroup of U_i/U_{i+1}.
(b) The group G_0/G_1 is cyclic and isomorphic to a subgroup of the group of roots of unity in the residue field of L. Its order is prime to p.
(c) The quotients G_i/G_{i+1} for $i \ge 1$ are abelian groups and are direct products of cyclic groups of order p. The group G_1 is a p-group.
(d) The group G_0 is the semi-direct product of a cyclic group of order prime to p with a normal subgroup whose order is a power of p.
(e) The groups G_0 and G are both solvable.

The GAP package [7] in Sage [13] can be used to find possible Galois groups as described for extensions of $\mathbb{Q}_p$ in [2–4]. For small degrees, the online L-functions and Modular Forms Database (LMFDB) [1] can also be used to find possible Galois groups with the necessary properties. The same technique in finding the Galois group for the p-adic case can be applied to the function field case. Consider one of the case 2 extensions. As mentioned above, one can use the methods described in [5, p. 587] to efficiently find a defining polynomial for K/F. For example, we find that $x^5 + 2x + 1$ is a defining polynomial for K/F in the case $p = 3$. By Theorem 4.10 defining polynomials for the two non-isomorphic case 2 extensions are given by $x^2 - T$ and $x^2 - \zeta T$ where T is a uniformizer in F and consequently a uniformizer for K/F and ζ is a primitive $p^5 - 1$-st root of unity. We will use Lemma 5.1 to discuss the properties of the Galois group and find the Galois group for a case 2 extension with $x^2 - T$ being a defining polynomial for L/K.

The Galois group of L/K is a solvable subgroup of S_n, or in this case S_{10}. There are 24 solvable subgroups of S_{10}. The Galois group will have a subfield corresponding to G/G_0, the Galois group of the unramified intermediate extension. This G/G_0 must be isomorphic to $\mathbb{Z}/5\mathbb{Z}$ since the Galois group of an unramified extension is always isomorphic to $\mathbb{Z}/f\mathbb{Z}$. From Lemma 5.1 part (a), G_0/G_1 is isomorphic to $\mathrm{Aut}(L/K)$ which is necessarily isomorphic to $\mathbb{Z}/2\mathbb{Z}$ since L/K is a degree two extension. Note that $\mathbb{Z}/2\mathbb{Z}$ is cyclic and of order prime to 5. In this particular case, since G_i is isomorphic to the trivial group for $i \geq 1$, $G_0 \cong G_0/G_1$. Thus the Galois group must have a normal subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The only group which fits these criteria is $\mathbb{Z}/10\mathbb{Z}$. Below is a table listing the Galois groups for all six degree 10 extensions:

Case	e	f	$\mathrm{Gal}(L/F)$
1	1	10	$\mathbb{Z}/10\mathbb{Z}$
2	2	5	$\mathbb{Z}/10\mathbb{Z}$
2	2	5	$\mathbb{Z}/10\mathbb{Z}$
3	5	2	F_5
4	10	1	$F_5 \times \mathbb{Z}/2\mathbb{Z}$
4	10	1	$F_5 \times \mathbb{Z}/2\mathbb{Z}$

Note that F_5 is the Frobenius group of order 20 which is isomorphic to a semidirect product $\mathbb{Z}/5\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z} \ltimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$.

The same methods of finding the Galois group of L/F can be applied to intermediate extensions. The following table contains information about the intermediate unramified and totally tamely ramified extensions in the case that $p = 3$.

Case	e	f	$\mathrm{Gal}(K/F)$	Polynomial for K/F	$\mathrm{Gal}(L/K)$	Polynomial for L/K
1	1	10	$\mathbb{Z}/10\mathbb{Z}$	$x^{10} + 2x^2 + 1$		
2	2	5	$\mathbb{Z}/5\mathbb{Z}$	$x^5 + 2x + 1$	$\mathbb{Z}/2\mathbb{Z}$	$x^2 - T$
2	2	5	$\mathbb{Z}/5\mathbb{Z}$	$x^5 + 2x^4 + 2x + 2$	$\mathbb{Z}/2\mathbb{Z}$	$x^2 - \zeta_{242} T$
3	5	2	$\mathbb{Z}/2\mathbb{Z}$	$x^2 + x + 2$	F_5	$x^5 - T$
4	10	1			$F_5 \times \mathbb{Z}/2\mathbb{Z}$	$x^{10} - T$
4	10	1			$F_5 \times \mathbb{Z}/2\mathbb{Z}$	$x^{10} - \zeta_2 T$

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