A Topological Structure on Certain Initial Algebras *

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Abstract

There is a well known natural topology on the set of compatible total orders on a group. A similar notion is defined on the set of distinct monomial algebras in polynomial and Laurent polynomial rings. We study the later topological structure for monomial algebras that come from rings of multiplicative invariants and show that they are either finite discrete spaces or homeomorphic to the Cantor set. This result agrees with several recent results on the space of left orders on a group.

1 Introduction

A group G is called left (respectively, right) orderable if there is a total order $\succcurlyeq$ on G compatible with the group operation, that is $\mathbf{a} \succcurlyeq \mathbf{b}$ implies $\mathbf{ca} \succcurlyeq \mathbf{cb}$ ($\mathbf{ac} \succcurlyeq \mathbf{bc}$)

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for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in G$. Not every group admits such an order; for example, finite groups do not. The study of orderable groups is not new, but in recent years there has been a considerable attention to the field due to the discovery of deep connections with topology, and dynamics of group action on a circle. Orderability is also a proven useful tool in 3-manifold theory. For details and other recent developments on orderable groups we refer to [1, 3, 6, 11]. An example of orderable group is the free abelian group $\mathbb{Z}^n$ with the lexicographic (or dictionary) order, $\succcurlyeq_{\text{lex}}$, given by $\mathbf{a} \succ_{\text{lex}} \mathbf{b}$ if an only if the first nonzero entry of $\mathbf{a}-\mathbf{b}$ is positive, for $\mathbf{a} \neq \mathbf{b} \in \mathbb{Z}^n$. Our investigation in this paper is based on the set of all compatible orders on $\mathbb{Z}^n$, denoted Ω. Sikora [11] defined a natural topology on Ω with subbasis $U_{\mathbf{a},\mathbf{b}} = \{\succcurlyeq \in \Omega \mid \mathbf{a} \succ \mathbf{b}\}$ for each $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ and showed that this space is a totally disconnected, compact metric space with no isolated points. Hence it is homeomorphic to the Cantor set. Linnell [6] also showed that the space of left orders on any orderable group is either finite or uncountable.

On the other hand, consider the Laurent polynomial ring, $\Bbbk[\mathbf{x}^{\pm 1}] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, in n variables. A topological structure on the set

$$\mathbb{V} = \{V : \ V \text{ is a } \Bbbk\text{−subspace of } \Bbbk[\mathbf{x}^{\pm 1}] \ \text{ spanned by monomials}\}$$

is defined by Kuroda [5]. We will present details in Section 2 below. Now let $\succcurlyeq \in \Omega$, the initial exponent of a nonzero polynomial $f \in \Bbbk[\mathbf{x}^{\pm 1}]$ with respect to $\succcurlyeq$ is

$$\text{in}_{\succcurlyeq}(f) = \max_{\succcurlyeq}\{\mathbf{a} \in \mathbb{Z}^n : \mathbf{x}^{\mathbf{a}} \text{ occurs in } f \text{ with nonzero coefficient}\}.$$

The initial algebra, $\text{in}_{\succcurlyeq}(R)$, of a subalgebra R of $\Bbbk[\mathbf{x}^{\pm 1}]$ with respect to $\succcurlyeq \in \Omega$ is the monomial algebra $\text{in}_{\succcurlyeq}(R) = \Bbbk[\mathbf{x}^{\text{in}_{\succcurlyeq}(f)} : f \in R \setminus \{0\}]$. Note that $\text{in}_{\succcurlyeq}(R) \in \mathbb{V}$. Like initial ideals in Gröbner basis theory, initial algebras play an important role for subalgebras due to the SAGBI theory pioneered by Robbiano and Sweedler [10] and independently by Kapur and Madlener [4]. The term “SAGBI” is an acronym for “Subalgebra Analogue to Gröbner Bases for Ideals.”

Consider a subgroup $\mathcal{G}$ of the symmetric group S_n and let R be the subring of polynomial or Laurent polynomial rings which are fixed under the action of $\mathcal{G}$ by permuting variables. Such rings are called permutation invariants. For the subalgebra R of permutation invariants, Kuroda [5] showed that the set of distinct initial algebras $\{\text{in}_{\succcurlyeq}(R) : \succcurlyeq \in \Omega\}$ is either finite or uncountable. Kuroda's argument is a careful analysis of the subspace topology $\{\text{in}_{\succcurlyeq}(R) : \succcurlyeq \in \Omega\} \subseteq \mathbb{V}$. Tesemma [13] generalized Kuroda's result on the cardinality of distinct initial algebras for subalgebras R that are invariant under action of an arbitrary subgroup of $\text{GL}_n(\mathbb{Z})$. These subrings are called multiplicative invariants. Note that permutations can be realized as permutation matrices, so multiplicative invariants are a larger class of invariant rings and include permutation invariants. Tesemma's approach did not use a topology on initial algebras but rather counted a family of convex polyhedral cones associated to these initial algebras.

In this paper we study $\{\mathrm{in}_{\succcurlyeq}(R) : \succcurlyeq \in \Omega\} \subseteq \mathbb{V}$ as a quotient space of Ω for an arbitrary ring of multiplicative invariants and show the following analogous result to the topology on Ω.

Theorem 1.1. *Let R be a subalgebra of multiplicative invariants of $\Bbbk[\mathbf{x}^{\pm}]$. The set of distinct initial algebras $\{\mathrm{in}_{\succcurlyeq}(R) : \succcurlyeq \in \Omega\}$ under the quotient topology from Ω is either a finite discrete space or homeomorphic to the Cantor set.*

2 Basics on Ring of Multiplicative Invariants

Let $\mathcal{G}$ be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and $\Bbbk[\mathbf{x}^{\pm 1}]$ be the Laurent polynomial ring in n variables. $\mathcal{G}$ acts on the multiplicative group of monomials $\{\mathbf{x}^{\mathbf{a}},\ \mathbf{a} \in \mathbb{Z}^n\}$ by $\varphi(\mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\varphi \cdot \mathbf{a}}$ where $\varphi \cdot \mathbf{a}$ is multiplication of the $n \times n$ matrix φ in $\mathcal{G}$ by the vector $\mathbf{a}$ in $\mathbb{Z}^n$. This action extends to the Laurent polynomial ring $\Bbbk[\mathbf{x}^{\pm 1}]$ by $\Bbbk$-algebra automorphism.

The subalgebra

$$\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}} = \{f \in \Bbbk[\mathbf{x}^{\pm 1}] \mid \varphi(f) = f,\ \forall\ \varphi \in \mathcal{G}\}$$

is called an algebra or ring of multiplicative invariants. The orbit sum of each monomial, $\vartheta(\mathbf{x}^{\mathbf{a}}) = \sum_{\varphi \in \mathcal{G}} \mathbf{x}^{\varphi \mathbf{a}}$, is an invariant polynomial. Moreover, the set

$$\{\vartheta(\mathbf{x}^{\mathbf{a}}) : \mathbf{a} \in \mathbb{Z}^n\}$$

is a $\Bbbk$-basis of the ring of multiplicative invariants. For details on the theory of multiplicative invariants, we direct the reader to [7].

Definition. Let $\varphi \in \mathrm{GL}_n(\mathbb{Z})$ be a linear transformation on $\mathbb{R}^n$. We call φ a *reflection* if φ fixes a hyperplane and $\varphi^2 = 1$, where 1 is the identity matrix. A group $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ is called a *reflection group* if it is generated by reflections.

The existence of a finite SAGBI basis for a ring of multiplicative invariants is tied to the group $\mathcal{G}$ acting on the algebra being a reflection group. For details on this fact we refer to [8]. In this paper, whether or not $\mathcal{G}$ is a reflection group gives distinct topological spaces of initial algebras.

3 Topological Structure on Ω and Initial Algebras

In this section we define a topological structure on initial algebras of a ring of multiplicative invariants R of a fixed finite subgroup $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ and show that the space of initial algebras is a compact subspace of the Cantor set.

Consider the canonical map from the topological space Ω to the set $\mathbb{V}$

$$\mathscr{Q}_R : \Omega \to \mathbb{V} : \succcurlyeq \mapsto \mathrm{in}_{\succcurlyeq}(R).$$

Let Ψ be the quotient space induced by $\mathscr{Q}_R$ on $\mathscr{Q}_R(\Omega)$, the set of initial algebras of R. Consider the equivalence relation $\sim_R$ on Ω defined by $\succcurlyeq \sim_R \succcurlyeq'$ if and only if R has the same initial algebra with respect to $\succcurlyeq$ and $\succcurlyeq'$. By definition

$$\Omega/{\sim_R} \cong \Psi$$

as topological spaces, where we give $\Omega/{\sim_R}$ the quotient topology corresponding to the equivalence relation. Because the equivalence classes that comprise $\Omega/{\sim_R}$ are analogous to the Gröbner regions of an ideal, the topological space Ψ with this quotient topology is a natural object of study as both a space of initial algebras and a space arising from a natural equivalence relation on Ω that captures the initial algebra structure of R.

Part of our main result is that if $\mathcal{G}$ is not generated by reflections, Ψ is homeomorphic to the Cantor set. By our definition of Ψ, were the map $\mathscr{Q}_R$ guaranteed to be injective, a trivial consequence would be that Ψ is the Cantor set. In section 4, we give an example where $\mathcal{G}$ is not generated by reflections but $\mathscr{Q}_R$ is not injective.

We now show that Ψ is a compact subspace of the Cantor set. We begin by restricting a metric defined by Kuroda [5] on $\mathbb{V}$ to our space Ψ to show that Ψ is metrizable.

Lemma 3.1. *The space Ψ is metrizable.*

Proof. First define any map that induces a finite filtration on $\mathbb{Z}^n$ as follows:

$$\rho : \mathbb{Z}^n \to \mathbb{N},$$

such that $\rho^{-1}(s)$ is finite for each $s \in \mathbb{N}$. An example of such a ρ is

$$\rho : (a_1, \ldots, a_n) \mapsto 1 + \sum |a_i|.$$

Consider the metric on $\mathbb{V}$ defined by Kuroda [5] as

$$d_\rho(V,W) = \begin{cases} \frac{1}{r}, & \text{if } r = \max\{s \in \mathbb{N} \mid x^{\mathbf{a}} \in V \Leftrightarrow x^{\mathbf{a}} \in W \text{ for all } \mathbf{a} \in \rho^{-1}([s])\} \text{ exists} \\ 0, & \text{if no such } r \text{ exists,} \end{cases}$$

for any $V, W \in \mathbb{V}$, where $[s] = \{1, \ldots, s\}$. Let Ψ' be the topological space obtained by restricting the metric d_ρ to $\mathscr{Q}_R(\Omega) \subset \mathbb{V}$. We will show that $\Psi = \Psi'$.

We first show that the map $\mathscr{Q}_R : \Omega \to \Psi'$ is continuous. Let $\mathscr{Q}_R(\succcurlyeq) \in \Psi'$ and $\varepsilon > 0$. Let $r \in N$ be such that

$$\frac{1}{r} < \varepsilon.$$

For any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, consider the subbasic open set

$$U^{\succcurlyeq}_{\mathbf{a},\mathbf{b}} = \begin{cases} U_{\mathbf{a},\mathbf{b}}, & \text{if } \mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}} \\ U_{\mathbf{b},\mathbf{a}}, & \text{if } \mathbf{x}^{\mathbf{b}} \succ \mathbf{x}^{\mathbf{a}} \\ \Omega, & \text{if } \mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{b}}. \end{cases}$$

Because $\mathcal{G}$ is finite, for any $\mathbf{a} \in \mathbb{Z}^n$

$$U^{\succcurlyeq}_{\mathbf{a}} = \bigcap_{\varphi \in \mathcal{G}} U^{\succcurlyeq}_{\mathbf{a},\varphi(\mathbf{a})}$$

is open in Ω. Also for any $\succcurlyeq' \in U^{\succcurlyeq}_{\mathbf{a}}$, $\mathbf{x}^{\mathbf{a}} \in \mathscr{Q}_R(\succcurlyeq)$ if and only if $\mathbf{x}^{\mathbf{a}} \in \mathscr{Q}_R(\succcurlyeq')$. Because $\rho^{-1}([r])$ is finite

$$U^{\succcurlyeq}_{r} = \bigcap_{\mathbf{a} \in \rho^{-1}([r])} U^{\succcurlyeq}_{\mathbf{a}}$$

is open in Ω. Then if $\succcurlyeq' \in U^{\succcurlyeq}_{r}$,

$$d_\rho(\mathscr{Q}_R(\succcurlyeq'), \mathscr{Q}_R(\succcurlyeq)) \leqslant \frac{1}{r} < \varepsilon.$$

Therefore $\mathscr{Q}_R : \Omega \to \Psi'$ is continuous.

Now because Ω is compact and Ψ' is a metric space and hence Hausdorff, our map $\mathscr{Q}_R : \Omega \to \Psi'$ is closed. Thus because $\mathscr{Q}_R : \Omega \to \Psi'$ is continuous, closed and surjective, $\mathscr{Q}_R : \Omega \to \Psi'$ is a quotient map. Therefore $\Psi = \Psi'$, so Ψ is metrizable. $\square$

Lemma 3.2. *The space Ψ is compact.*

Proof. By definition $\mathscr{Q}_R : \Omega \to \Psi$ is continuous. Therefore because Ω is compact and Ψ is metrizable, Ψ is compact. $\square$

Lemma 3.3. *The space Ψ is totally disconnected.*

Proof. The metric for Ψ constructed in the proof of Lemma 3.1 takes only rational values. $\square$

Proposition 3.4. *The space Ψ is a compact subset of the Cantor set.*

Proof. The proposition follows from Lemmas 3.1, 3.2, 3.3. $\square$

The rest of the paper is devoted to prove that the space Ψ is also perfect for ring of invariants under action of a finite non-reflection group. First, we need to further investigate a certain subset of Ω.

4 Orders on $\mathbb{Z}^n$ Determined by Weight Vectors

Robbiano [9] gave a classification of all compatible orders on $\mathbb{Z}^n$ from the point of view of computational commutative algebra. This classification plays an important role in Gröbner and SAGBI basis theories. In this section we consider those orders defined by a single weight vector $\mathbf{w} \in \mathbb{R}^n$. First note that the map $\mathbf{a} \mapsto \mathbf{w} \cdot \mathbf{a}$ is one-to-one if and only if $\mathbf{w}$ is of rational dimensional n, i.e. the coordinates of $\mathbf{w}$ span an n-dimensional vector space over $\mathbb{Q}$. It follows that each vector $\mathbf{w}$ of rational dimension n determine an ordering, $\succcurlyeq_{\mathbf{w}}$, on $\mathbb{Z}^n$ by

$$\mathbf{a} \succcurlyeq_{\mathbf{w}} \mathbf{b} \;\Leftrightarrow\; \mathbf{a} \cdot \mathbf{w} \geqslant \mathbf{b} \cdot \mathbf{w}, \quad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$$

It is immediately apparent from the definition that $\succcurlyeq_{\mathbf{w}} = \succcurlyeq_{\lambda\mathbf{w}}$ for $\lambda \in \mathbb{R}_{>0}$. But if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two non-parallel vectors then the set $\{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w}_1 > 0 > \mathbf{v} \cdot \mathbf{w}_2\}$ is a non-empty open convex cone in $\mathbb{R}^n$, and hence by density of rationals contains some $\mathbf{a} \in \mathbb{Z}^n$. It follows that $\mathbf{a} \succcurlyeq_{\mathbf{w}_1} 0$ and $0 \succcurlyeq_{\mathbf{w}_2} \mathbf{a}$ and hence $\succcurlyeq_{\mathbf{w}_1} \neq \succcurlyeq_{\mathbf{w}_2}$. Thus any order induced by a weight vector is induced by a unique vector in the set

$$\mathcal{S} := \{\mathbf{w} \in \mathbb{R}^n : ||\mathbf{w}|| = 1 \wedge \dim_{\mathbb{Q}}(\mathbf{w}) = n\}$$

where $\dim_{\mathbb{Q}}$ denotes the rational dimension.

Proposition 4.1 (Kuroda [5])**.** *If we endow $\mathcal{S}$ with the subspace topology from the standard metric topology on $\mathbb{R}^n$, then the map*

$$\iota : \mathcal{S} \to \Omega : \mathbf{w} \mapsto \succcurlyeq_{\mathbf{w}}$$

is continuous.

Proof. Consider a subbasic open set

$$U_{\mathbf{a},\mathbf{b}} = \{\succcurlyeq \in \Omega \mid \mathbf{a} \not\succcurlyeq \mathbf{b}\}.$$

Its preimage is

$$\begin{aligned} \iota^{-1}(U_{\mathbf{a},\mathbf{b}}) &= \{\mathbf{w} \in \mathcal{S} \mid \mathbf{w}\cdot\mathbf{a} > \mathbf{w}\cdot\mathbf{b}\} = \{\mathbf{w} \in \mathcal{S} \mid \mathbf{w}\cdot(\mathbf{b}-\mathbf{a}) > 0\} \\ &= \mathcal{S} \cap \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}\cdot(\mathbf{b}-\mathbf{a}) > 0\} \end{aligned}$$

which is an open set in $\mathcal{S}$. $\square$

Kuroda [5] also showed that the image $\iota(\mathcal{S})$ is dense in Ω. His proof relies on the classification of orders on $\mathbb{Z}^n$ by Robbiano in [9]. We give an alternative proof using some facts from convex polyhedral geometry.

Definition.

1. A subset C of $\mathbb{R}^n$ is called a *polyhedral cone* if there is a finite subset $X \subseteq C$ such that
$$C = \operatorname{Cone}(X) = \left\{ \sum_{\mathbf{v}\in X} \lambda_{\mathbf{v}}\mathbf{v} \;\middle|\; \lambda_{\mathbf{v}} \in \mathbb{R}_{\geqslant 0} \right\}$$
2. The *dual* of a polyhedral cone C is the set $C^\vee = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w}\cdot\mathbf{v} \geqslant 0 \;\forall\; \mathbf{v} \in C\}$
3. A cone C is *strongly convex* if $\{0\}$ is a face of C.

Lemma 4.2 (Cox, et.al. [2, Chapter 1])**.** *Let C be a polyhedral cone. The following are equivalent:*

i. *C is strongly convex.*

ii. *C contains no positive-dimensional subspaces of $\mathbb{R}^n$.*

iii. $\dim \sigma^\vee = n$.

Lemma 4.3 (Cox, et.al. [2, Chapter 1])**.** *Let $X \subseteq \mathbb{Z}^n$ be finite and $C = \operatorname{Cone}(X)$. Then*

$$C \cap \mathbb{Q}^n = \left\{ \sum_{\mathbf{v}\in X} \lambda_{\mathbf{v}}\mathbf{v} \;\middle|\; \lambda_{\mathbf{v}} \in \mathbb{Q}_{\geqslant 0} \right\}.$$

Proposition 4.4. *The image $\iota(\mathcal{S})$ is dense in Ω.*

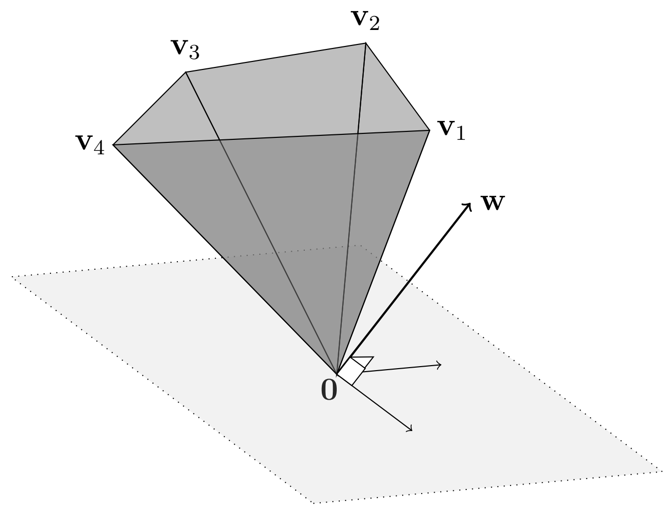

Figure 1: Depiction of how $\mathbf{w} \in \iota^{-1}(U)$ corresponds to the geometry of the cone C when $U \neq \emptyset$

Proof. Consider a nonempty basic open set

$$U = \bigcap_{i=1}^{k} U_{\mathbf{a}_i, \mathbf{b}_i}.$$

Let $\mathbf{v}_i = \mathbf{a}_i - \mathbf{b}_i \in \mathbb{Z}^n$ and $C = \operatorname{Cone}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

First we show that C is strongly convex. Suppose to the contrary that C is not strongly convex. Then by fact 4.2, C contains a positive-dimensional subspace of $\mathbb{R}^n$. Thus there exists $\{r_i\}_i, \{s_i\}_i \subset \mathbb{R}_{\geqslant 0}$ such that

$$0 \neq \sum_{i=1}^{k} r_i \mathbf{v}_i = -\sum_{i=1}^{k} s_i \mathbf{v}_i.$$

Without loss of generality assume $s_1 \neq 0$. Then we can solve for $\mathbf{v}_1$ to get that $-\mathbf{v}_1 \in C$. Then by Lemma 4.3,

$$-\mathbf{v}_1 \in C \cap \mathbb{Q}^n = \left\{ \sum_{\mathbf{v} \in S} \lambda_{\mathbf{v}} \mathbf{v} \mid \lambda_{\mathbf{v}} \in \mathbb{Q}_{\geqslant 0} \right\}. \tag{1}$$

For any $\succcurlyeq \in U$, each $\mathbf{v}_i \succ 0$, but by (1) $-\mathbf{v}_1 \succ 0$ so $\mathbf{v}_1 \prec 0$. Thus no such $\succcurlyeq$ exists, contradicting that U is nonempty.

Therefore C is strongly convex. Thus by Proposition 4.2 above, $\dim C^\vee = n$ and hence there exists $\mathbf{w} \in C^\vee \cap \mathcal{S}$. Then for all $i \in \{1, \ldots, k\}$, $\mathbf{w} \cdot \mathbf{v}_i > 0$.

Thus $\iota(\mathbf{w}) \in U$, and $\iota(\mathcal{S})$ is dense in Ω. $\square$

We now use weight orders to give an example of a non-reflection group for which $\mathscr{Q}_R$ is not injective.

Example. Consider the subgroup

$$\mathcal{G} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \{1, \varphi\} \subset \mathrm{GL}_3(\mathbb{Z})$$

and its ring of multiplicative invariants $R = \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]^{\mathcal{G}}$.

Consider $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} x_3^{a_3} \in \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ and $\mathbf{w} = (w_1, w_2, w_3) \in \mathcal{S}$

The monomials $\mathbf{x}^{\mathbf{a}}$ and $\varphi(\mathbf{x}^{\mathbf{a}}) = x_1^{-a_1} x_2^{-a_2} x_3^{a_3}$ have the same coefficient in any polynomial in R. Thus $\mathbf{x}^{\mathbf{a}} \in \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$ if and only if $\mathbf{x}^{\mathbf{a}} \succcurlyeq_{\mathbf{w}} \varphi(\mathbf{x}^{\mathbf{a}})$, which occurs if and only if

$$w_1 a_1 + w_2 a_2 + w_3 a_3 \geqslant -w_1 a_1 - w_2 a_2 + w_3 a_3.$$

Therefore $\mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$ depends only on w_1 and w_2, so $\mathscr{Q}_R$ is not injective.

5 Perfectness of the Space $\mathscr{Q}_R(\Omega)$

In this section we will prove that if $\mathcal{G}$ is not generated by reflections then Ψ is perfect. In [5], Kuroda proves a similar result for the special case when $\mathcal{G}$ is a subgroup of the symmetric group S_n that is not generated by transpositions. Our proof is inspired by Kuroda's, but requires important changes to work in our more general setting.

In our setting the action of $\mathcal{G}$ can change the length of exponent vectors. Thus where Kuroda uses a polytope

$$M = \left\{ (v_1, \cdots, v_n) \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \sum_{i=1}^n v_i = 1 \right\}$$

whose surface is closed under permutations, we use the existence of a $\mathcal{G}$-invariant inner product to construct a surface that is closed under the action of $\mathcal{G}$.

Let $I_\sigma := \ker(\sigma - 1)$ where $\sigma \in \mathrm{GL}_n(,\mathbb{Z})$ and 1 is the identity matrix. Similarly for $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ let $I_\mathcal{G} := \bigcup_{\sigma \in \mathcal{G}\setminus\{1\}} I_\sigma$.

Definition & **Remarks.** Let $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$.

(i) An inner product $\langle \cdot, \cdot \rangle_\mathcal{G}$ on $\mathbb{R}^n$ is called *G-invariant* if $\langle \mathbf{x}, \mathbf{y} \rangle_G = \langle \varphi(\mathbf{x}), \varphi(\mathbf{y}) \rangle_\mathcal{G}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$ and $\varphi \in \mathcal{G}$. Such inner product exists for finite groups by defining a new inner product as follows:

$$\langle \mathbf{x}, \mathbf{y} \rangle_\mathcal{G} := \sum_{\varphi \in \mathcal{G}} \varphi(\mathbf{x}) \cdot \varphi(\mathbf{y}).$$

(ii) We define a *G-sphere*, $S^G := \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle_G = 1\}$.

(iii) A point $\mathbf{x} \in S^\mathcal{G}$ is said to be *projectively rational* if $\mathbf{x} = c\mathbf{r}$ for some $c \in \mathbb{R}$ and $\mathbf{r} \in \mathbb{Q}^n$. It is clear from density of $\mathbb{Q}^n$ on $\mathbb{R}^n$ that projectively rational points are dense in S^G.

Lemma 5.1. *For any $\sigma \in G$, $\dim I_\sigma = n - 1$ if and only if σ is a reflection.*

Proof. The forward direction is given by the definition of a reflection.
Now suppose $\dim I_\sigma = n - 1$; that is, σ fixes a hyperplane. Since σ preserves angles in the G-invariant inner product, it preserves the line orthogonal to I_σ, and hence must map this line onto itself or its opposite. In the first case, σ is the identity, which contradicts that $\dim I_\sigma < n$. Thus σ must negate this line, so it is a reflection. □

In particular, I_σ disconnects S^G if and only if σ is a reflection.

Remark 5.2. Let $\tau \in G$. Then τ maps I_G onto I_G.

Proof. For any $\sigma \in G$, if $\mathbf{x} \in I_\sigma$, $\sigma(\mathbf{x}) = \mathbf{x}$. Then

$$\tau\sigma\tau^{-1}(\tau(\mathbf{x})) = \tau\sigma(\mathbf{x}) = \tau(\mathbf{x})$$

so $\tau(\mathbf{x}) \in I_{\tau\sigma\tau^{-1}}$. Thus τ maps I_σ onto $I_{\tau\sigma\tau-1}$, and hence maps I_G onto I_G. □

Lemma 5.3. *Assume G is not a reflection group. Then for every $\mathbf{a} \in S^G \setminus I_G$, every connected component of $S^G \setminus I_G$ contains at least two points of the orbit*

$$\vartheta(\mathbf{a}) = \{\sigma(\mathbf{a}) \mid \sigma \in G\}.$$

Proof. First, note that each $\sigma \in G$ maps S^G into S^G because σ preserves the G-invariant inner product.
By Remark 5.2, any $\sigma \in G$ preserves I_G, so it must map $S^G \setminus I_G$ onto itself. Since σ is a homeomorphism, it preserves components of $S^G \setminus I_G$.
If $S^G \setminus I_G$ has only one connected component, that component must contain $\vartheta(\mathbf{a})$. Since $\mathbf{a} \notin I_G$, $\vartheta(\mathbf{a})$ contains at least two elements, satisfying the claim.
Otherwise, let $C \neq D$ be two connected components of $S^G \setminus I_G$. Consider a path

$$\psi : [0,1] \to S^G$$

between a point in C and a point in D. Further suppose that this path does not intersect any I_σ for non-reflections σ or any $I_\sigma \cap I_\tau$ for $\sigma \neq \tau \in G$ and that

$$\{t \in [0,1] \mid \psi(t) \in I_\tau \text{ for some reflection } \tau \in G\}$$

is finite. Denote these points $t_0 < \cdots < t_r$, and for each t_i let τ_i be the reflection in G whose invariant subspace contains $\psi(t_i)$. Since each τ_i is a homeomorphism, $\tau_r \circ \cdots \circ \tau_1$ is a bijection mapping C onto D. Since $\tau_r \circ \cdots \circ \tau_1$ is also a bijection on $\vartheta(\mathbf{a})$, C and D must contain the same cardinality of elements of $\vartheta(\mathbf{a})$.
Now, suppose some component C has only one element of $\vartheta(\mathbf{a})$; then every component does. For each $\sigma \in G$, $\sigma(\mathbf{a}) \neq \mathbf{a}$ because $\mathbf{a} \notin I_\sigma$. Hence $\sigma(\mathbf{a})$ lies in a different component D. Since $\sigma(\mathbf{a})$ is D's unique element of $\vartheta(\mathbf{a})$, the product of reflections τ mapping C onto D must also map $\mathbf{a}$ onto $\sigma(\mathbf{a})$. Then $\sigma\tau^{-1}$ fixes $\mathbf{a} \notin I_G$, so we must have $\sigma\tau^{-1} = I$, and hence $\sigma = \tau$. Thus σ is a product of reflections in G. Hence G is a reflection group, contradicting the assumption. □

Lemma 5.4. *For any $\mathbf{a}, \mathbf{b} \in S^G \setminus I_G$, if there is a path from $\mathbf{a}$ to $\mathbf{b}$ in S^G, there is such a path in which projectively rational points are dense.*

Proof. Let our original path be given by $\gamma_0 : [0,1] \to S^G \setminus I_G$, with $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$.

For each $t \in [0,1]$, fix

$$\delta(t) = \inf\{d(\gamma_0(t), \mathbf{y}) \mid, \mathbf{y} \in I_G\} > 0.$$

We now describe another path γ_1 from $\mathbf{a}$ to $\mathbf{b}$:

> Set $\varepsilon_1(t) = \frac{\delta}{2}$. By Proposition 4.4 we can find a projectively rational point ρ with $d(\rho, \gamma_0(\frac{1}{2})) < \varepsilon(t)$. Now we take γ_1 to be another continuous function with $\gamma_1(0) = \gamma_0(0)$ and $\gamma_1(1) = \gamma_0(1)$, but $\gamma_1(\frac{1}{2}) = \rho$; restrict γ_1 to satisfy $d(\gamma_1(t), \gamma_0(t)) < \varepsilon(t)$ for any $t \in [0,1]$.

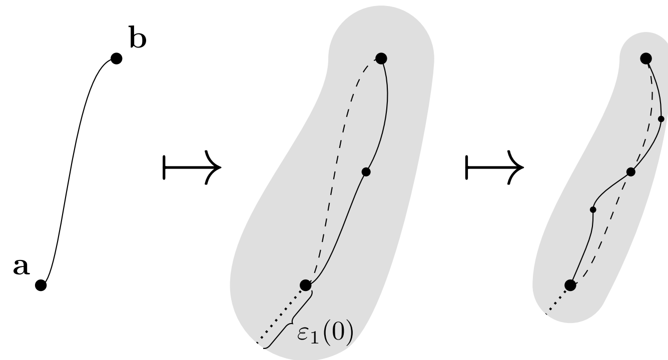

Figure 2: Example γ_0, γ_1, and γ_2.

For each i, we define γ_i from γ_{i-1} similarly, by repeating this procedure on each interval of the form $[\frac{j}{2^{i-1}}, \frac{j+1}{2^{i-1}}]$; however, we modify it so that the $\varepsilon_i(t)$ used is

$$\varepsilon_i(t) = \min\left\{\frac{1}{2}\left(\inf\{d(\gamma_{i-1}(t), \mathbf{y}) \mid \mathbf{y} \in I_G\} - \frac{\delta(t)}{4}\right), \frac{1}{2^i}\right\}.$$

This $\varepsilon_i(t)$ ensures that $d(\gamma_i(t), \gamma_{i+1}(t)) < \frac{1}{2^i}$ for all $t \in [0,1]$, so the sequence $\gamma_0, \gamma_1, \ldots$ is uniform convergent. Then it has a limit γ, which is continuous and has image in $S^G \setminus I_G$ still since at each t $\gamma(t)$ has distance to any point in I_G at least $\frac{\delta(t)}{4}$. Since

$$\gamma_i\left(\frac{m}{2^i}\right) = \gamma_{i+k}\left(\frac{m}{2^i}\right)$$

is projectively rational for any $i, k, m \in \mathbb{N}$, it follows that γ is projectively rational at any dyadic fraction; but dyadic fractions are dense in $[0,1]$, so projectively rational points are dense in $\gamma([0,1])$. $\square$

Remark 5.5. Let $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$ and the coordinates of $\mathbf{w}$ be rationally independent. Then for any nonempty open $I \subset \mathbb{R}$, there exists $\delta \in I$ such that the coordinates of $\mathbf{w} - \delta\mathbf{v}$ are rationally independent.

Proof. Let $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. Let $\delta \in \mathbb{R}$ such that it is not in $\mathbb{Q}[w_1, \ldots, w_n, v_1, \ldots, v_n]$, the extension of $\mathbb{Q}$ by adjoining the coordinates of $\mathbf{w}$ and $\mathbf{v}$. We claim that the coordinates of $w - \delta v$ are rationally independent. Suppose to the contrary that there exist $a_1, \ldots, a_n \in \mathbb{Q}$, not all zero, such that

$$\sum_{i=1}^{n} a_i(w_i - \delta v_i) = 0.$$

Then because the w_i are rationally independent $\sum_{i=1}^n a_i w_i \neq 0$. Thus $\sum_{i=1}^n a_i v_i \neq 0$ so

$$\delta = \frac{\sum_{i=1}^n a_i w_i}{\sum_{i=1}^n a_i v_i} \in \mathbb{Q}[w_1, \ldots, w_n, v_1, \ldots, v_n],$$

which contradicts our choice of δ. Since any nonempty open $I \subset \mathbb{R}$ contains such a δ, this completes our proof. $\square$

Theorem 5.6. *Let R be a ring of multiplicative invariants under action of a non-reflection subgroup of $\mathcal{G} \leqslant \mathrm{GL}(n, \mathbb{Z})$. For any $\mathbf{w} \in \mathcal{S}$, any open ball $B(\mathbf{w}, \varepsilon) \subseteq \mathcal{S}$ contains $\mathbf{w}_\delta$ such that $\mathscr{Q}_R(\succcurlyeq_{\mathbf{w}_\delta}) \neq \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$.*

Proof. Fix arbitrary $\mathbf{w} \in \mathcal{S}$ and let $\succcurlyeq \, = \, \succcurlyeq_{\mathbf{w}} = \iota(\mathbf{w})$. Now choose $\mathbf{a} \in S^G \setminus I_G$ for which $\mathbf{a} \succcurlyeq \sigma(\mathbf{a})$ for any $\sigma \in G$. By Lemma 5.3, we can find $\sigma(\mathbf{a})$ in the same component for some $\sigma \in G$. Then we can define a path between them, that is, a continuous $\gamma : [0,1] \to S^G \setminus I_G$ with $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \sigma(\mathbf{a})$. By Lemma 5.4, we can choose γ to have projectively rational points dense in $\gamma([0,1])$.
Now fix $\tau = \sigma^{-1}$, and observe that

$$t \mapsto \mathbf{w} \cdot (\gamma(t) - \tau(\gamma(t)))$$

is a continuous function. Since $\mathbf{a} \succ \tau(\mathbf{a})$ and $\sigma(\mathbf{a}) \prec \tau(\sigma(\mathbf{a})) = \mathbf{a}$, this function is positive for $t = 0$ and negative for $t = 1$; by the intermediate value theorem, it must have at least one zero in $(0,1)$. Now, for each $\rho \in G$, we define

$$T_\rho = \{t \in [0,1] \mid \mathbf{w} \cdot (t - \rho(t)) = 0\}$$

and

$$T = \bigcup_{\rho \in G \setminus \{I\}} T_\rho.$$

We know T_τ is nonempty and thus T is, so we define $t_0 = \inf T$ and $\mathbf{b} = \gamma(t_0)$. Each T_ρ is the preimage of 0 under a continuous map, so it is closed; then T is a finite union of closed sets and hence also closed. Thus $t_0 \in T$, so for some $\rho \in G$

$$\mathbf{w} \cdot (\mathbf{b} - \rho(\mathbf{b})) = 0,$$

but since $\mathbf{b} \notin I_\rho$,

$$\mathbf{b} - \rho(\mathbf{b}) \neq 0.$$

Moreover, for all $t < t_0$ and all $\phi \in G$,

$$\mathbf{w} \cdot (\gamma(t) - \phi(\gamma(t)) > 0. \tag{2}$$

Next we take a sequences of projectively rational points $\{t_i\}_{i>0}$ in $[0,1]$ such that each $t_i < t_0$, but the sequence converges to t_0. Define $\mathbf{a}_i = \gamma(t_i)$.
Now, because $\mathbf{a}_i - \rho(\mathbf{a}_i)$ approaches $\mathbf{b} - \rho(\mathbf{b})$, and its dot product with $\mathbf{w}$ approaches 0 from above, for each ε' we can find $N_{\varepsilon'}$ such that for $i > N_{\varepsilon'}$

$$\big|(\mathbf{b} - \rho(\mathbf{b})) \cdot \big((\mathbf{b} - \rho(\mathbf{b}) - (\mathbf{a_i} - \rho(\mathbf{a_i}))\big)\big| < \varepsilon' \tag{3}$$

and

$$0 < \mathbf{w} \cdot (\mathbf{a}_i - \rho(\mathbf{a}_i)) < \varepsilon'. \tag{4}$$

For each $\delta > 0$ define $\mathbf{w}_\delta = \mathbf{w} - \delta(\mathbf{b} - \rho(\mathbf{b}))$. For any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\left| \mathbf{w} - \frac{\mathbf{w}_\delta}{|\mathbf{w}_\delta|} \right| < \varepsilon$$

and by Lemma 5.5 we can choose δ such that $\mathbf{w}_\delta$ is rationally independent. For such δ set

$$\varepsilon' = \frac{\delta}{1+\delta} |\mathbf{b} - \rho(\mathbf{b})|^2.$$

Using (3) and (4), we have for large enough i

$$\begin{aligned}
\mathbf{w}_\delta \cdot (\rho(\mathbf{a}_i) - \mathbf{a}_i) &= \big(\mathbf{w} - \delta(\mathbf{b} - \rho(\mathbf{b}))\big) \cdot (\rho(\mathbf{a}_i) - \mathbf{a}_i) \\
&= \mathbf{w} \cdot (\rho(\mathbf{a}_i) - \mathbf{a}_i) \\
&\quad - \delta(\mathbf{b} - \rho(\mathbf{b})) \cdot \big((\rho(\mathbf{a}_i) - \mathbf{a}_i) - (\rho(\mathbf{b}) - \mathbf{b}) + (\rho(\mathbf{b}) - \mathbf{b})\big) \\
&> -\varepsilon' - \delta\varepsilon' + \delta \left|\mathbf{b} - \rho(\mathbf{b})\right|^2 \\
&= (-1-\delta)\varepsilon' + (1+\delta)\varepsilon' = 0
\end{aligned}$$

Now defining

$$\mathbf{w}'_\delta = \frac{\mathbf{w}_\delta}{|\mathbf{w}_\delta|} \in \mathcal{S}$$

we see $\mathbf{w}_\delta \cdot (\rho(\mathbf{a}_i) - \mathbf{a}_i) > 0$ as well, that is, $\rho(\mathbf{a}_i) \succ_{\mathbf{w}'_\delta} \mathbf{a}_i$. Since $\mathbf{a}_i$ is psuedorational, there is a real r such that $\mathbf{a}'_i = r\mathbf{a}_i \in \mathbb{Z}^n$, and

$$\rho(\mathbf{a}'_i) = \rho(r\mathbf{a}_i) = r\rho(\mathbf{a}_i) \succ_{\mathbf{w}'_\delta} \mathbf{a}'_i.$$

Now any G-invariant polynomial containing $\mathbf{x}^{\mathbf{a}'_i}$ must also contain $\mathbf{x}^{\rho(\mathbf{a}'_i)}$, so $\mathbf{x}^{\mathbf{a}'_i}$ cannot be its initial term; thus $\mathbf{x}^{\mathbf{a}'_i} \notin \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}'_\delta})$.
But the orbit sum

$$f(\mathbf{x}) = \sum_{\rho \in G} \mathbf{x}^{\rho(\mathbf{a}'_i)}$$

is a G-invariant polynomial with initial term $\mathbf{x}^{\mathbf{a}'_i}$ under under $\succcurlyeq_{\mathbf{w}}$ by (2), so $\mathbf{x}^{\mathbf{a}'_i} \in \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$.
Thus $\mathscr{Q}_R(\succcurlyeq_{\mathbf{w}}) \neq \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}'_\delta})$, and by construction $\mathbf{w}'_\delta \in B(\mathbf{w}, \varepsilon)$. $\square$

Remark 5.7. Tesemma has a similar result in [13, Theorem 3.5], but his approach doesn't make use of a topology on Ω. Moreover, his result is for orders induced by weight vectors in $\mathbb{R}^n$ with a fixed "tie breaker".

6 Main Results

Let $R \subseteq \Bbbk[\mathbf{x}^{\pm 1}]$ be the ring of multiplicative invariants as before under action of a finite group $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$.

Remark 6.1. If $\mathcal{G}$ is a reflection group then by [13, Theorem 1.1] the space $\mathscr{Q}_R(\Omega)$ is finite. Since Ω is metrizable, is a finite discrete space.

Proposition 6.2. *If $\mathcal{G}$ is a non-reflection subgroup of $\mathrm{GL}(n, \mathbb{Z})$, $\mathscr{Q}_R(\Omega)$ is perfect.*

Proof. Let $A \in \mathscr{Q}_R(\Omega)$ be an initial algebra.

Case 1 If $A = \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$ for some rationally independent $\mathbf{w}$, then any open set containing $\mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$ has open preimage under $\mathscr{Q}_R \circ \iota$. By Lemma 5.6, the preimage has an element $\mathbf{w}_\delta$ satisfying $\mathscr{Q}_R(\iota(\mathbf{w}_\delta)) \neq \mathscr{Q}_R(\iota(\mathbf{w}))$. Hence the open set contains the distinct element $\mathscr{Q}_R(\iota(\mathbf{w}_\delta))$.

Case 2 If $A = \mathscr{Q}_R(\succcurlyeq)$ but $A \neq \mathscr{Q}_R(\succcurlyeq_{\mathbf{w}})$ for any rationally independent $\mathbf{w}$, then by the density of weight orders any neighborhood of A contains a weight order, which cannot be A.

In either case, A is a limit point, so all points of $\mathscr{Q}_R(\Omega)$ are limit points. $\square$

Propositions 3.4 and 6.2 prove the main result:

Theorem 6.3. *If $\mathcal{G}$ is a non-reflection subgroup of $\mathrm{GL}_n(, \mathbb{Z})$, then Ψ is homeomorphic to the Cantor set.*

7 Acknowledgements

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8 The Aftermath: Invariants of Conjugates of the Alternating Group

Let $\mathcal{G}$ be a subgroup of S_n acting on the polynomial ring $\Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ by permuting the variables, and let $R = \Bbbk[\mathbf{x}]^{\mathcal{G}}$. Recall the ring of invariants of $\mathcal{G}$ has a finite SAGBI basis in any order if and only if $\mathcal{G}$ is generated by transpositions. A corollary of this follows.

Theorem 8.1 (Göbel)**.** *The ring of invariants of the alternating group $A_n \leqslant S_n$ has no finite SAGBI bases.*

Göbel studied the existence of a matrix $\sigma \in \mathrm{GL}_n(\Bbbk)$ such that the ring of invariants of conjugates of the alternating group $\sigma A_n \sigma^{-1} = \{\sigma\varphi\sigma^{-1} : \varphi \in A_n\}$ have a finite SAGBI bases under a special linear action, where $\sigma(f)(\mathbf{x}) = f(\sigma\mathbf{x})$. If we have such a matrix, we can determine subalgebra membership by determining if elements' conjugates lie in the conjugate ring of invariants. For the last two weeks, we turned our attention to the open problem posed by Göbel in which he wanted to find all such matrices. To begin tackling this problem we began calculating the desired matrices in Sage using the subduction algorithm described in Sturmfels [12]. The Sage code we used is given below.

```
singular.eval("LIB \"sagbi.lib\";")
#Use sage to run the subduction algorithm
#to check if the first set all reduce
#to constant polynomials by the second
def reducesToConstant(P,I):
        if P[0]==0: return true
        PP = singular(Ideal(P))
        II = singular(Ideal(I))
        R = singular.sagbiReduce(PP,II).sage()
        for r in R.gens():
                if not r.is_constant(): return false
        return true

#Compute the generators of the toric ideal associated to matrix A
#Then compose them with the n polynomials I
def ToricGens(A,I): return map(lambda x:x(I.gens()),ToricIdeal(A).gens())

def leading_exp(f): return vector(f.lm().exponents()[0])

#Compute the matrix defining the relevant toric ideal
```

```
def toricMatrix(I): return matrix(map(leading_exp ,I.gens())).transpose()

#determine if the set I is a SAGBI basis
def isSagbi(I): return reducesToConstant(ToricGens(toricMatrix(I),I),I)

def sympoly(R,i):
        n = R.ngens()
        if i==n+1:
                prod=1
                for a in xrange(n):
                        for b in xrange(a):
                                prod *= (R.gens()[b]-R.gens()[a])
                return prod
        else:
                f = Symmentricfunctions(R).elementary()[i]
                return f.expand(n)(R.gens())

#return the basis for the alternating group on the variables of R
def sympolys(R): return [sympoly(R,i) for i in xrange(1,R.ngens()+2)]

#the linear action of D on the polynomial f
def linear_act(D,f): return f((D * vector(f.variables())).list())

#the basis of the form tested by Goebel:
#D applied to the basis for the alternating group
def goebel_basis(D,R): return map(lambda f:linear_act(D,f),sympolys(R))

def scan(n,h,output='matrices.txt',ord='lex'):
        f = file(output,'w')
        R = PolynomialRing(QQ,['x'+str(x) for x in xrange(n)],order=ord)
        for M in MatrixSpace(GF(2*h+1),n):
               MM = M.apply_map(lambda x:ZZ(x)-h)
                if MM.determinant()==0: continue
                if isSagbi(Ideal(goebel_basis(MM,R))):
                        f.write(str(MM)+'\n\n')
```

Running `sage(3,1)` shows that there are 1776 3-dimensional matrices with entries in $\{-1,0,1\}$. Below we post a subset of these matrices.

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

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