REU Kings Write-up

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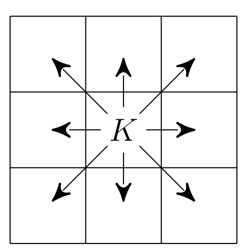
Abstract

Let $f_{m,n}$ be the number of placements of non-attacking kings on a $m \times n$ board. We will analyze the generating function $f_{m,n}(x)$ for placing kings using matrices, Lagrange interpolation, Markov chains, and Zeckendorff representations. We will find bounds for $f_{m,n}^{\frac{1}{mn}}(x)$ and find recurrence relations for its series expansion.

1 Introduction

On a $m \times n$ board, a king can attack all of the cells adjacent to it. Our main goal is to find the total number of placements of non-attacking kings on such a board, and we will then analyze the corresponding generating function and other characteristics of the board placements.

We define $f_{m,n}$ to be the number of placements of non-attacking kings on a $m \times n$ board. The cells that a king can attack are displayed in the diagram below.



Consider the configurations on a $1 \times n$ board.

We build the $1 \times n$ boards by adding a 1×2 block $\square\boxed{K}$ to the end of the $1 \times (n-2)$ boards or by adding an empty cell $\square$ to the end of the $1 \times (n-1)$ boards.

$$
\begin{aligned}
f_{1,0} &= 1 \\
f_{1,1} &= 1+1=2 \\
f_{1,2} &= 1+2=3 \\
f_{1,3} &= 2+3=5 \\
f_{1,4} &= 3+5=8 \\
f_{1,5} &= 5+8=13 \\
f_{1,6} &= 8+13=21
\end{aligned}
$$

Theorem 1.1. *For $n \geq 0$, $f_{1,n} = F_{n+2}$, where F_n is the n^{th} Fibonacci number.*

Proof. The proof is by simple induction by building the $1 \times n$ boards from the $1 \times (n-1)$ boards and the $1 \times (n-2)$ boards as described above. $\square$

We can also show that $f_{2,n} = J_{n+2}$, where J_n is the n^{th} Jacobsthal number by the same method.

2 Generating Functions

The generating function for the $m \times n$ board is

$$f_{m,n}(x) = \sum_{i=0}^{\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil} a_i x^i,$$

where a_i, the coefficient of x^i, is the number of ways of placing i kings.

For example, the five placements ($f_{1,3} = 5$) of kings on the 1×3 board are below.

K		

K		K

	K	

		K

So the generating function for these placements is $f_{1,3}(x) = 1 + 3x + x^2$.

We observe that

$$f_{1,n}(x) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n-i+1}{i} x^i.$$

Then the coefficient of x^i is the number of ways of placing i kings on the $1 \times n$ board. . If we place i kings, then there will be $n-i$ empty tiles. Consider the tiles below to be these $n-i$ tiles in a row.

$$\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}\ldots\boxed{}$$

Place kings in i of the $(n-i+1)$ spaces in between and on either side of the empty tiles. Each king must be separated from every other king by at least one tile, so we will choose i of the $(n-i+1)$ spaces between the tiles and place the i kings in those spaces.

Below, see an example of placing kings in the spaces between the tiles.

$$\boxed{K}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{K}\,\boxed{}\,\boxed{K}\,\boxed{}\,\boxed{}\,\boxed{}\ldots\boxed{}\,\boxed{K}$$

Then we find that the coefficient of x^i is

$$\binom{n-i+1}{i}.$$

We can show that $f_{2,n}(x)$ has a similar closed form.

$$f_{2,n}(x) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n-i+1}{i} 2^i x^i.$$

In this equation, we will have $n-i$ 2×1 columns, $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$, instead of $(n-i)$ empty tiles. Choose i of the $(n-i+1)$ spaces between the empty columns to be a column containing a king. Within each column, there can be at most one king. Within the column, the king can be placed in one of the 2 cells, either the top or the bottom cell, which gives 2^i ways to place the i kings within the columns after the columns have been chosen.

Closed formulas for larger boards, $3 \times n$ and larger have not been found.

2.1 Recursive Generating Functions

Since no closed form has been found for boards larger than $2 \times n$, we turn to defining the boards recursively.

2.1.1 The $2 \times n$ Board Recursive Definiton

Consider the $2 \times n$ board. The possible endings of the $2 \times n$ board are

$$\begin{array}{cccc} \text{Column} & 1 & 2 & 3 \\ & \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} & \begin{array}{|c|}\hline \\ \hline K \\ \hline\end{array} & \begin{array}{|c|}\hline K \\ \hline \\ \hline\end{array}. \end{array}$$

Then

$$f_{2,n}(x) = \sum_i x^{\text{\#kings in column i}} G_i(x).$$

where $G_i(x)$ is the generating function for placing kings in the remainder of the board that ends with column i.

The recursive generating function is built by considering the three possible endings of the board.

- If the board ends in $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, then we can have any of the $2 \times (n-1)$ placements before this column and $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ contains 0 kings, so $G_1(x) = f_{2,n-1}(x)$ and this ending contributes $x^0 f_{2,n-1}(x)$ to the sum.

- If the board ends in $\begin{array}{|c|}\hline \ \\ \hline K \\ \hline\end{array}$, then the column before must be empty, $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\begin{array}{|c|}\hline \ \\ \hline K \\ \hline\end{array}$ and we can have any of the $2 \times (n-2)$ placements before this empty column. As $\begin{array}{|c|}\hline \ \\ \hline K \\ \hline\end{array}$ contains 1 king, we have $G_2(x) = f_{2,n-2}(x)$ and this ending contributes $x^1 f_{2,n-2}(x)$ to the sum.

- If the board ends in $\begin{array}{|c|}\hline K \\ \hline \ \\ \hline\end{array}$, then the column before must be empty, $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\begin{array}{|c|}\hline K \\ \hline \ \\ \hline\end{array}$, and we can have any of the $2 \times (n-2)$ placements before this empty column. As $\begin{array}{|c|}\hline K \\ \hline \ \\ \hline\end{array}$ contains 1 king, we have $G_3(x) = f_{2,n-2}(x)$ and this ending contributes $x^1 f_{2,n-2}(x)$ to the sum.

Then for $n \geq 2$,

$$\begin{aligned} f_{2,n}(x) &= x^0 f_{2,n-1}(x) + x^1 f_{2,n-2}(x) + x^1 f_{2,n-2}(x) \\ &= f_{2,n-1}(x) + 2x f_{2,n-2}(x). \end{aligned}$$

2.1.2 Recursive Generating Functions for $1 \times n$, $2 \times n$, ..., and $6 \times n$ boards.

We can similarly define recursive generating functions for larger boards. The recursions become more complicated, so we find them by solving a system of equations using coefficients of x^k in the generating functions, described later in this section. The recursive definitions for the $1 \times n, 2 \times n, \ldots,$ and$6 \times n$ are listed below.

$$
\begin{aligned}
\text{For } n \geq 2, \quad f_{1,n}(x) &= f_{1,n-1}(x) + x f_{2,n-2}(x) \\
\text{For } n \geq 2, \quad f_{2,n}(x) &= f_{2,n-1}(x) + 2x f_{2,n-2}(x) \\
\text{For } n \geq 3, \quad f_{3,n}(x) &= (1+x) f_{3,n-1}(x) + (2x + x^2) f_{3,n-2}(x) + (-x^2 - x^3) f_{3,n-3}(x) \\
\text{For } n \geq 4, \quad f_{4,n}(x) &= (1+x) f_{4,n-1}(x) + (3x + 4x^2) f_{4,n-2}(x) + (x^2 - 3x^3) f_{4,n-3}(x) \\
&\quad + (-3x^4) f_{4,n-4}(x) \\
\text{For } n \geq 7, \quad f_{5,n}(x) &= (1+2x) f_{5,n-1}(x) + (3x + 8x^2 + 3x^3) f_{5,n-2}(x) + (-10x^3 - 5x^4) f_{5,n-3}(x) \\
&\quad + (-2x^3 - 12^4 - 11^5 - 3^6) f_{5,n-4}(x) + (9x^5 + 15x^6 + 4x^7) f_{5,n-5}(x) \\
&\quad + (-1x^7 + 3x^8 + x^9) f_{5,n-6}(x) + (-3x^9 - x^{10}) f_{5,n-7}(x) \\
\text{For } n \geq 8, \quad f_{6,n}(x) &= (1+2x) f_{6,n-1} + (4x + 15x^2 + 11x^3) f_{6,n-2} + (3x^2 - 4x^3 - 16x^5) f_{6,n-3} \\
&\quad + (-x^3 - 30x^4 - 65x^5 - 42x^6) f_{6,n-4} + (5x^5 + 42x^6 + 38x^7) f_{6,n-5} \\
&\quad + (6x^7 + 48x^8 + 62x^9) f_{6,n-6} + (-18x^9 - 24x^{10}) f_{6,n-7} \\
&\quad + (-8x^{11} - 24x^{12}) f_{6,n-8}
\end{aligned}
$$

Starting with the $3 \times n$ recursive definition and continuing for the recursive definitions for larger boards, negative correction terms start to appear, the depth of the recurrences and the degree of the polynomial with each term increase. These increments are not linear.

Some patterns occur in the recursive generating functions. The coefficients in the polynomials before the recursion terms occur at the same positions in each recursion.

In $f_{1,n}(x), f_{2,n}(x), \ldots, f_{6,n}(x)$, the polynomials preceding the $f_{i,n-1}(x)$ and the $f_{i,n-2}(x)$ terms have positive coefficients.

In $f_{4,n}(x)$, $f_{5,n}(x)$, and $f_{6,n}(x)$, the polynomials preceding the $f_{i,n-3}(x)$ and the $f_{i,n-4}(x)$ terms have negative coefficients.

In $f_{5,n}(x)$ and $f_{6,n}(x)$, the polynomials preceding the $f_{i,n-5}(x)$ and the $f_{i,n-6}(x)$ have positive coefficients and the $f_{i,n-7}(x)$ and $f_{i,n-8}(x)$ (the $(n-8)$ term is only in $f_{6,n}(x)$) have negative coefficients.

This suggests that for every pair of terms in the recursive generating functions $f_{i,n}(x)$, we alternate between positive terms and negative correction terms for all i.

2.1.3 Method for Finding Recursive Generating Functions

The recursive generating funtions for the $1 \times n$ and $2 \times n$ boards were found by building the boards using the ending columns.

We solved the recursive generating functions for the $m \times n$ boards for $m \in \{3, 4, 5, 6\}$ using a system of equations to solve for the polynomials before each term in the recursion.

For each m we consider the equation,

$$f_{m,n}(x) = a_1(x)f_{m,n-1}(x) + a_2(x)f_{m,n-2}(x) + \dots a_k(x)f_{m,n-k}(x)$$

where

$$a_i(x) = a_{i,0} + a_{i,1}x + a_{i,2}x^2 + \dots + a_{i,j}x^j$$

for some degree j and each $a_{i,l}$ is a constant.

We start with a guess for the number of terms in the recursion, k, and a guess for the degree of each polynomial $a_i(x), j$. For n and t sufficiently large we then examine the coefficient of x^t in the above equation and vary t until we have enough information to solve for each $a_{i,l}$ in each polynomial $a_i(x)$, which will be integers when the system is solved correctly. The solutions should be integers so if the system does not have integer solutions, we increase k, the depth of the recursion we are testing, and increase j, the degree of each polynomial $a_i(x)$ until each $a_{i,l}$ is an integer in the solution.

This method works in theory for any m, but for $m > 6$, solutions could not be found in a reasonable time.

3 Adjacency Matrices

Let G be a graph on vertices $1, 2, 3, \cdots, F_{m+2}$ with adjacency matrix $A_n = (a_{i,j})$ where

$$a_{ij} = 1 \text{ if there is an edge from } i \text{ to } j.$$

Then we can count the number of walks in G by computing powers of A_m.

The number of walks of length 1 from i to j is $a_{ij} = (A_m)_{ij}$.

Then

$$a_{i,j} = \begin{cases} 0 & , \quad \text{if there is a path from } i \text{ to } j \\ 1 & , \quad \text{otherwise} \end{cases}$$

Here, the vertices $1, 2, 3, \cdots, F_{m+2}$ represent the F_{m+2} possible placements of kings on $m \times 1$ columns and adjacency between vertices i and j means that the ith column and the jth column can be adjoined such that there are no attacking kings. Recall that F_m denotes the m^{th} Fibonacci number.

Observe that the matrix A_m is symmetric, since if column i can be adjacent to column j, then column j can be adjacent to column i. Thus A_m^k is also symmetric for every k.

By the same argument, using induction over k we obtain that $a_{ij}^{(k)} = (A_m^k)_{ij}$ is the number of walks of length k from i to j.

The adjacency matrix for the $1 \times n$ board is below. $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The adjacency matrix for the $2 \times n$ board is below. $A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The adjacency matrix for the $3 \times n$ board is below. $A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 3.1. $A_{m+1} = \begin{pmatrix} A_m & A_{m-1} \\ A_{m-1} & 0 \end{pmatrix}$.

Proof. This proof follows by induction and uses the same idea of building $1 \times n$ boards using $1 \times (n-1)$ boards with $\square$ added to the end of each and using $1 \times (n-2)$ boards with $\square\boxed{K}$ added to the end of each. For this proof we consider building $(m+1) \times 1$ columns with $m \times 1$ columns and $(m-1) \times 1$ columns and considering their adjacencies. $\square$

For $k = 2$,

$$a_{ij}^{(2)} = \sum_{k=1}^{F_{n+1}} a_{ik} a_{kj}$$

, where $(A_n^2)_{ij} = a_{ij}^{(2)}$. Now, $a_{ik} = 1$ if the ith column can be followed by the kth column and $a_{kj} = 1$ if the kth column can be followed by the jth column. Thus, $a_{ik}a_{kj} = 1$ if there is a path of length 2 between the ith column and the jth column. Each 1 in the sum of $a_{ij}^{(2)}$ corresponds to a path of length 2 from the ith column to the jth column, so $a_{ij}^{(2)}$ is the number of paths of length 2 from the ith column to the jth column. Similarly for paths of length n.

Theorem 3.2. *$f_{m,n} = I^t A_m^{n-1} I$ where $I^t = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$.*

For example, the number of placements on the $3 \times n$ board, $f_{3,n}$ is

$f_{3,n} = I^T (A_3)^{n-1} I$, where $I^T = (1,1,1,1,1)$ and $A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$,

the adjacency matrix of the $3 \times n$ board.

3.0.4 Generating functions using Matrices

Let $A_m(x) = X_m A_m$, where A_m is the adjacency matrix for the $(m \times 1)$ board and

$$X_m = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x^{a_1} & 0 & 0 & 0 \\ 0 & 0 & x^{a_2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & x^{a_i} \end{pmatrix}$$

where a_i is the number of kings on the i^{th} $(m \times 1)$ board.

Theorem 3.3. *The general formula for the generating function is* $f_{m,n}(x) = v^T A_m(x)^{n+1} v$, *where* $v^T = (1, 0, 0, \cdots, 0)$.

This means that the generating function $f_{m,n}(x)$ is the top left entry of the matrix $A_m(x)$ raised to the power $n+1$.

For example, the generating function for the $3 \times n$ board is

$$f_{3,n}(x) = v^T A_3(x)^{n+1} v, \text{ where } v^T = (1,0,0,0,0) \text{ and } A_3(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ x & 0 & 0 & x & 0 \\ x & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A_3(x)$ is the adjacency matrix of the $3 \times n$ board

3.0.5 Symmetrizing the Adjacency Matrix

Let $u = v = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$ and consider A_m and X as defined above. We can rewrite the generating function for the $m \times n$ board as

$$f_{m,n}(x) = u^t X^{\frac{1}{2}} (X^{\frac{1}{2}} A_m X^{\frac{1}{2}})^n X^{-\frac{1}{2}} v$$

Here, $X^{\frac{1}{2}}$ is obtained as a "square root" of X in matrix terms, since $X$is a diagonal matrix. We let $x = t^2$ and replace x by t in $X^{\frac{1}{2}}$. Now look at the matrix raised to the nth power.

$$\text{Let } X^{\frac{1}{2}} = T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & t^{a_{2,2}} & 0 & \dots & 0 \\ 0 & 0 & t^{a_{3,3}} & \dots & 0 \\ \vdots & 0 & 0 & t^{a_{k,k}} & 0 \\ 0 & 0 & 0 & \dots & t^{a_{m,m}} \end{pmatrix}$$

The reason that we choose to work with $X^{\frac{1}{2}}$ and analyze $X^{\frac{1}{2}} A_m X^{\frac{1}{2}}$ is that the later matrix is symmetric, while $X_m A_m$ is not. Since we are analyzing the eigenspace of the power of this matrix, we need it to be symmetric for the eigenvalues and the entries of the eigenvectors to have real values. Below are the eigenvalues for the power of the symmetrized adjacency matrix, from which we looked for patterns or recursions, since they have been used in the previous papers to compute the entropy, η.

Adjacency Matrices for ($1 \times n$) :

Below is the symmetrized adjacency matrix: TA_1T raised to several powers of n. We computed powers of TA_1T to see if there was any pattern to the eigenvalues as we increase n. We hope to correlate the eigenvalues to the entropy constants as we increase n in this way. Note that for $n = 1$, the largest eigenvalue is the golden ratio when $t = 1$.

Table 1: Eigenvalues for Powers of the Symmetrized Matrix of A_1

n	$(TA_1T)^n$	Eigenvalues
1	$\begin{pmatrix} 1 & t \\ t & 0 \end{pmatrix}$	$\pm\frac{1}{2}\sqrt{4t^2+1}+\frac{1}{2}$
2	$\begin{pmatrix} t^2+1 & t \\ t & t^2 \end{pmatrix}$	$\pm\frac{1}{2}\sqrt{4t^2+1}+t^2+\frac{1}{2}$
3	$\begin{pmatrix} 2\,t^2+1 & t^3+t \\ t^3+t & t^2 \end{pmatrix}$	$\frac{3}{2}t^2 \pm \frac{1}{2}(t^2+1)\sqrt{4t^2+1}+\frac{1}{2}$
4	$\begin{pmatrix} t^4+3t^2+1 & 2t^3+t \\ 2t^3+t & t^4+t^2 \end{pmatrix}$	$t^4+2t^2 \pm \frac{1}{2}(2t^2+1)\sqrt{4t^2+1}+\frac{1}{2}$
5	$\begin{pmatrix} 3t^4+4t^2+1 & t^5+3t^3+t \\ t^5+3t^3+t & 2t^4+t^2 \end{pmatrix}$	$\frac{5}{2}t^4+\frac{5}{2}t^2 \pm \frac{1}{2}\sqrt{4t^2+1}(t^4+3t^2+1)+\frac{1}{2}$
6	$\begin{pmatrix} t^6+6t^4+5t^2+1 & 3t^5+4t^3+t \\ 3t^5+4t^3+t & t^6+3t^4+t^2 \end{pmatrix}$	$t^6+\frac{9}{2}t^4+3t^2 \pm \frac{1}{2}\sqrt{4t^2+1}(3t^4+4t^2+1)+\frac{1}{2}$

No pattern was clear between the powers of TA_1T, so we looked at the eigenvalues of A_i for i ranging from 1 to 6. From past research we know that the largest eigenvalue will tell us the entropy constant when we raise it to certain powers, but, we wanted to look at these eigenvalues with the variable t still in them. After $m = 2$ the eigenvalues do not appear to follow a pattern.

We notice that the generating function for the Catalan numbers is one of the eigenvalues for TA_1T. For $T(A_1)^2T$, replacing t with $\sqrt{2t}$ gives a similar generating function.

So far we also know that $f_{m,n}(x) = A_m^{n+1}(x)[0,0]$ (the first entry of the power matrix). Another idea is to take the generating function of the power matrix. We obtain

$$(\sum_{n\geq 0} A_m^{n+1}(x)z^n)[0,0] = \sum_{n\geq 0} f_{m,n}(x)z^n.$$

Here we add a variable z. By the geometric series formula, the left hand side is equal to $(I - A_m(x)z)^{-1}[0,0]$. If we can compute this inverse in a reasonable time, then we can expand it in series and the coefficients will be exactly $f_{m,n}(x)$, the values that we need for the double interpolation. We observed that the inverse is equal to

$$\frac{det(cofactor)}{det(I - A_m(x)z)} = \frac{poly(x,z)}{poly(x,z)}.$$

Computing the cofactor and the denominator matrix is efficient in $Sage$, but computing their determinants is very inefficient, even for small values of k. However, we expanded the denominator determinant and we obtained the recursive formula for $f_{k,n}(x)$ from the coefficients of z, for small values of k, which agree with the recurrence relations obtained previously.

4 Counting King Configurations With Trees and Zeckendorf Representation

The motivation behind finding a different way to calculate the total number of king configurations on an $m \times n$ board is that computing large powers of adjacency matrices is very inefficient. However, random boards can be generated more efficiently for larger m and n.

We use the tree theorem described in the next section and Zeckendorf's theorem to generate random $m \times n$ boards and approximate $f_{m,n}$.

4.1 The Tree Theorem

4.1.1 Estimating the Size of a Tree

We use the following procedure gives an estimation for the number of leaves of a tree.

1. We start at the root. Then we say that X_1 is the number of children of the root.
2. We choose one of the children uniformly. Then X_2 is the number of children of the child that we have chosen uniformly.
3. We repeat this process of choosing a child uniformly until we reach a leaf, that is when $X_n = 1$.

Then the expected number of leaves on the tree is

$$E(X) = \prod_{i=1}^{n} X_i.$$

4.1.2 Using the Tree Theorem to Count King Configurations

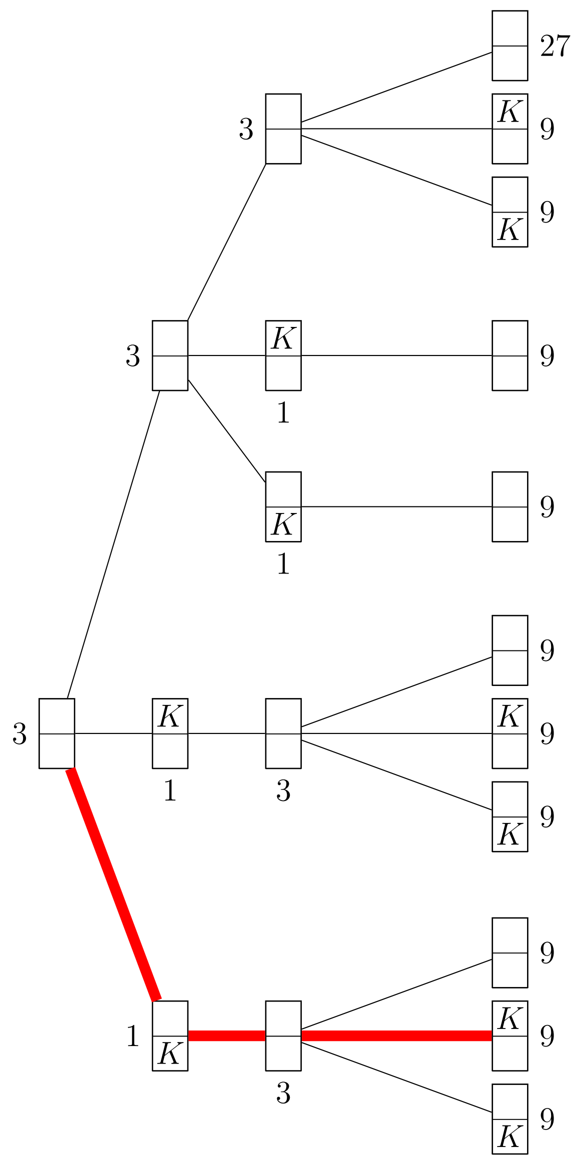
We will follow this procedure of estimating the number of leaves of a tree to approximate the number of configurations of kings on a $m \times n$ board.

The root of the tree will be an empty $m \times 1$ column, and the tree we are considering from the root is of depth $(n + 1)$, one larger than the actual size of the board we want to compute. This is because the rootis not counted in the random generation simply because it is always empty. Using the empty column for the root of the tree guarantees that the set of x_1 children of the root is equal to the set of possible $m \times 1$ columns, which means that the $m \times n$ board can start with any of the possible columns.

Each node in the tree represents a column in the $m \times n$ board. The children of any column node are the boards that can follow the column node. So, the leaves at the end of the tree are all the possible ways to add the last column to form each possible $m \times n$ board. To generate a random $m \times n$ board we follow a path from the root to a leaf node of depth $n + 1$ which will be the final column in the generated board.

4.1.3 Using the Tree Theorem to Count the 2×3 Configurations

The tree below corresponds to generating random 2×3 boards.



The 2×3 board generated by the highlighted path is:

		K
K		

We begin by noting that the root node is not part of the final chessboard generated by this tree. In this example there are 11 leaves at the end of the 11 paths, which correspond to the 11 possible 2×3 boards. In the highlighted path we have $E(X) = (3)(1)(3) = 9$. The values of $E(X)$ for other paths are listed to the right of the leaves. Notice that only one path has $E(X) = 27$, so in total the average is $\frac{27+9(10)}{11} \approx 10.6$, which is an approximation for 11, the total number of boards. Repeating the process of building a 2×3 board by following a path to a leaf 1000 times and averaging the values for $E(x)$ yields approximately 10.565.

To select the edge corresponding to the next column that we will add to the board randomly, we use the following theorem as a way to place kings on the column we are adding to the board.

4.2 Constructing Random Columns using Zeckendorf Representation

4.2.1 Introduction to Zeckendorf Representation

Theorem 4.1. *(Zeckendorf's Theorem). Every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers.*

Then we can write every integer N as a Zeckendorf sum, the sum of Fibonacci numbers, where we consider $\{F_n\}_{n\geq 1} = F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, \ldots$.

We can then respresent N as a binary string, which we will call the Zeckendorf representation of N, where the i^{th} digit in the string is equal to 1 if the Zeckendorf sum of N contains F_i and the i^{th} digit in the string is equal to 0 otherwise.

Consider the integer 102 written as a Zeckendorf sum.

$$\begin{aligned} 102 &= 89 + 34 + 8 + 3 \\ &= 0(1) + 0(2) + 1(3) + 0(5) + 1(8) + 0(13) + 0(21) + 1(34) + 0(55) + 1(89) \end{aligned}$$

The Zeckendorf sum for 102 corresponds to the Zeckendorf representation 0010100101.

We use the Zeckendorf representations to place kings on a board by replacing each 1 with a king and replacing each 0 with a blank tile. Since the 1's in the Zeckendorf representation are all not adjacent, this method ensures that we generate boards with non-attacking kings.

4.2.2 Creating Random Columns with Zeckendorf Representation

Recall that the number of placements of kings on a $1 \times m$ board, $f_{1,m}$ is equal to F_{m+2}. Also recall $f_{m,1} = f_{1,m} = F_{m+2}$. We will choose one of the placements uniformly, that is we will choose a random integer z such that $0 \leq z \leq f_{m,1} - 1 = F_{m+2}$ and map each z to a unique placement of kings.

We can generate random $m \times 1$ columns containing non-attacking kings using this the following method.

1. We will pick a number p uniformly in $[0, F_{n+2})$.

2. We write p in Zeckendorf representation, that is as a unique string of 1's and 0's, and add 0's to the end of the string has length m
3. We invert the binary string with length m so that it appears as a column and change the column string into a $m \times 1$ column by replacing each 1 with a king and each 0 with an empty tile.

This procedure maps a random integer in $[0, F_{n+2})$ to a placement of non-attacking kings on a $m \times 1$ column. Since every integer has a unique Zeckendorf representation, every integer in the interval maps to a distinct $m \times 1$ column. Notice that

$$|\{ p \mid p \in \mathbb{Z} \text{ and } p \in [0, F_{n+2})\}| = f_{m,1}$$

Thus the map between integers in the interval $[0, F_{n+2})$ and $m \times 1$ columns is bijective and we can generate all $m \times 1$ columns using this method.

4.2.3 Generating Random 10×1 Columns

Below are examples of random 10×1 columns generated by picking a random number $p \in [0, F_{12})$ and finding the 10×1 column containing non-attacking kings that p maps to. Note that $F_12 = 144$, so we are choosing p in the interval $[0, 144)$.

Below are 10×1 columns for $p = 8, 84, 46, 105, 37, 97, 59$. Each tile in the leftmost column in the following example indicates the Fibonacci number that corresponds to each tile in a 10×1 column and thus the Fibonacci number that each king placed in that row contributes.

	8	84	46	105	37	97	59
89				K		K	
55		K					K
34			K		K		
21		K					
13				K			
8	K	K	K			K	
5							
3			K	K	K		K
2							
1			K				K

For $p = 84$, we have $84 = 8 + 21 + 55$, so the kings placed in the column labeled 84 correspond to the Fibonacci numbers $8, 21$, and 55 in the leftmost column.

4.3 King Placements for Large Boards

The table below contains data collected in $Sage$ from building 10000 random boards for $n = 100$ and averaging this data. We vary m from 1 to 20, and let $m = 100$ in the last row and consider the average number of configurations in $m \times 100$ boards.

Table 2: Average Number of King Configurations on a $m \times n$ Board

m	$n = 100$
1	$9.27372692193079 \times 10^{20}$
2	$1.92508539129887 \times 10^{30}$
3	$7.87383482188875 \times 10^{44}$
4	$5.53837407771687 \times 10^{56}$
5	$7.46023597845611 \times 10^{69}$
6	$1.14621053623519 \times 10^{82}$
7	$3.48648536802097 \times 10^{95}$
8	$7.36266538738001 \times 10^{106}$
9	$2.04348416955687 \times 10^{120}$
10	$2.63382947497018 \times 10^{133}$
11	$6.06235489343007 \times 10^{146}$
12	$2.98740918782812 \times 10^{158}$
13	$6.54894969990404 \times 10^{170}$
14	$1.05730296438712 \times 10^{183}$
15	$5.07793708902948 \times 10^{194}$
16	$4.43787570609291 \times 10^{207}$
17	$2.71275700442780 \times 10^{220}$
18	$4.22004700158934 \times 10^{233}$
19	$1.13794977915193 \times 10^{245}$
20	$9.27527936214568 \times 10^{257}$

4.4 Average Number of Kings By Generating Random Boards

While collecting data for the number of configurations using Zeckendorf representations and the tree theorem, we average the number of kings on each random board generated to get an average number of kings on $m \times 100$ and $m \times 1000$ boards. This data is compared with the king densities in Markov chains.

Table 3: Average Number of Kings on a $m \times n$ Board

m	$n = 100$	$n = 1000$
1	27.8	275.29
2	37.26	371.01
3	59.17	586.01
4	72.68	720.05
5	91.54	910.10
6	106.70	1059.91
7	124.42	1236.44
8	141.12	1399.14
9	157.24	1566.90
10	174.28	1735.51
11	191.24	1899.69
12	207.55	2067.25
13	225.6	2233.57
14	240.79	2401.67
15	257.45	2564.83
16	275.20	2735.06
17	291.93	2898.21
18	307.81	3067.35
19	326.13	3233.92
20	341.90	3404.85
21	360.88	3571.54
100	1684.57	16734.31

5 Markov Chains and King Densities

When building boards using the Zeckendorf method, we found that only the current ending column was needed to build the next column in the board. Since that is the case, the process of building boards has the memoryless property, described in terms of the Markov property in this section.

5.1 Markov Chains and $m \times n$ Boards

Definition 5.1 (Markov Chain)**.** *Let $\{X_n\}_{n\geq 0} = X_0, X_1, X_2, \ldots$ be a sequence of discrete random variables taking values in a set S, called the state space, such that*

$$P(X_{n+1}|X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i_n) = P(X_{n+1} = j|X_n = i_n)$$

$\forall j, i_0, \ldots, i_n \in S$ and all n. Then $\{X_n\}_{n\geq 0}$ is a Markov chain.

We will consider the process of building $m \times n$ chess boards with non attacking kings as a Markov chain, where each possible column arrangement of kings on the $m \times 1$ board is a state in the state space S. If the current ending column of an $m \times (n-1)$ board is column i, then the probability of adding board j is

$$P(X_n = j | X_{n-1} = i) = \begin{cases} 0 & , \quad \text{columns } i, j \text{ cannot be adjacent} \\ \frac{1}{b} & , \quad \text{o.w., } b \text{ is the } number \text{ of columns that can be adjacent to column } i \end{cases}$$

5.1.1 The Markov Chain for the $3 \times n$ Board

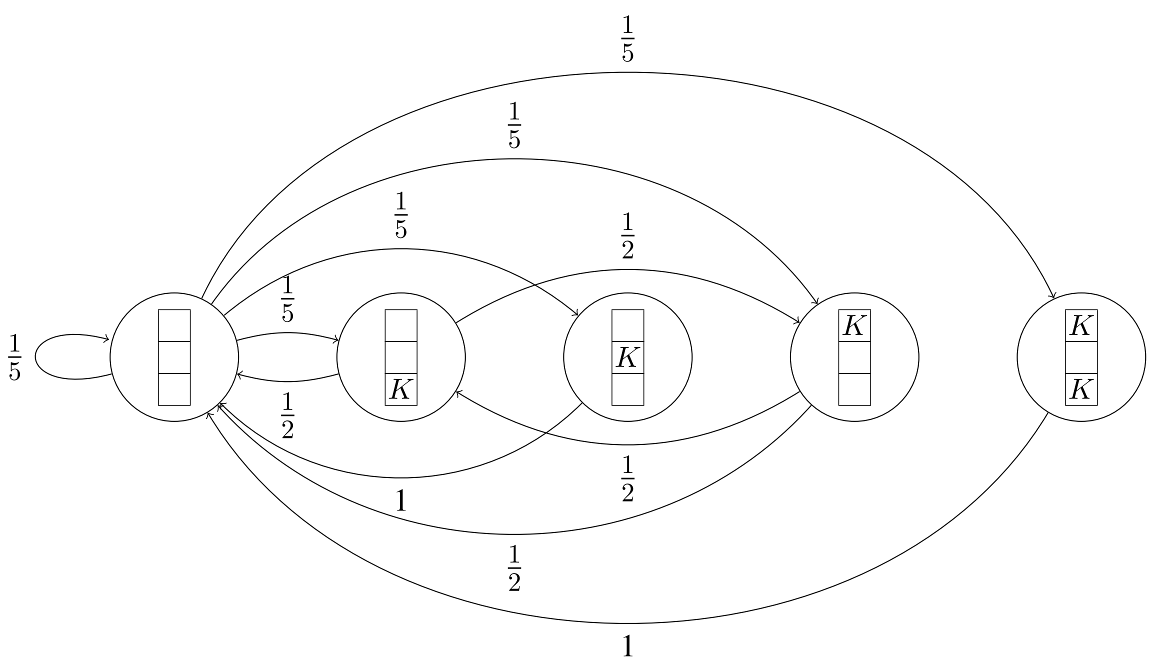
Recall that the adjacency matrix for the $3 \times n$ board is

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The transition matrix associated with the Markov chain for the $3 \times n$ board is

$$P_3 = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that P_3 is A_3 with row sums equal to 1. Let state i be the state associated with row i of the matrices A_3 and P_3. The transition graph associated with the Markov chain for the $3 \times n$ board is below.



We notice that this chain is finite, irreducible, and aperiodic. Thus a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ exists such that $\pi P_3 = \pi$ and $\sum_i \pi_i = 1$ and the subscripts in the π distribution vector, $1, 2, 3, 4,$ and 5, denote the states in the Markov chain and correspond to the 3×1 columns in the transition graph above. Furthermore, this stationary distribution is the limit distribution, representing the long term behaviour of the Markov chain.

For the Markov chain associated with the $3 \times n$ board we can solve for the stationary distribution to obtain

$$\pi = \left(\pi_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}, \pi_{\begin{smallmatrix}\square\\\square\\\boxed{K}\end{smallmatrix}}, \pi_{\begin{smallmatrix}\square\\\boxed{K}\\\square\end{smallmatrix}}, \pi_{\begin{smallmatrix}\boxed{K}\\\square\\\square\end{smallmatrix}}, \pi_{\begin{smallmatrix}\boxed{K}\\\square\\\boxed{K}\end{smallmatrix}}\right) = \left(\frac{5}{11}, \frac{2}{11}, \frac{1}{11}, \frac{2}{11}, \frac{1}{11}\right)$$

The stationary distribution gives the the proportion of columns in a $3 \times n$ board that we expect to see for large n. This means that we expect $\frac{5}{11}$ of the columns in a $3 \times n$ board to be the column $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, and we expect $\frac{1}{11}$ of the columns to be the column $\begin{smallmatrix}\boxed{K}\\\square\\\boxed{K}\end{smallmatrix}$.

5.2 Limit Distributions for Chess Boards

For all m the Markov chain for the $m \times n$ board will have finite state space. The chain will also be irreducible since all columns can reach all other columns, that is for any columns i and j, we can reach column j from column i in two steps by placing an empty column between columns i and j. The chain will be aperiodic since the empty column can follow itself.

Thus a unique stationary distribution π exists for all m. Furthermore, this stationary distribution is the limit distribution, representing the long term behaviour of the Markov chain. In general the limit distribution for the $m \times n$ board is

$$\pi_{m,n} = (\pi_1, \pi_2, \ldots, \pi_f)$$

where f denotes $f_{1,m} = F_{m+2}$, the number of placements of kings on the $1 \times m$ board. Each entry in the distribution π_i gives the proportion of time that column i appears in a $m \times n$ board for large n. Here

$$\pi_i = \frac{N_i}{N}$$

where N_i is the number of neighbors of column i, and N is the sum of all the N_i 's, equal to the sum of the entries of A_m or the number of placements of kings on the $m \times 2$ board. The number of neighbours of column i is equal to the number of columns that can be adjacent to column i.

In the $3 \times n$ example above, the column $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, which we will call state $i = 1$, has $N_1 = 5$ neighbours and $N = 11$ is the total number of neighbours of all of the 3×1 columns. Thus $\pi_1 = \frac{5}{11}$. For any sized board the sum of all π_i is 1, since we divide each N_i by the total number of neighbours,

$$\sum_{i \geq 1} \pi_i = \sum_{i \geq 1} \frac{N_i}{\sum\limits_{i \geq 1} N_i} = 1$$

5.3 Using Markov Chains to Find the Density of Kings of $m \times n$ Boards

Using this limit distribution and weight vectors that give the number of kings in each column we can find king densities $m \times n$ boards when n is large.

Table 5 contains the king densities, in kings per tile, of $m \times n$ boards where we vary m. The Markov king density of each $m \times n$ board, denoted ρ_m, was found using the weight vectors and the limiting distribution of the Markov chain with state space equal to the set of $m \times 1$ columns. The Zeckendorf densities were calculated using the average number of kings in one hundred $m \times 1000$ random Zeckendorf boards.

Table 4: Comparing Markov King Densities and Zeckendorf King Densities

m	Markov King Density, ρ_m	Zeckendorf Densities, $m \times 1000$
1	0.333333333333333	0.275290000000000
2	0.200000000000000	0.185505000000000
3	0.212121212121212	0.195336666666667
4	0.190476190476190	0.180012500000000
5	0.190697674418605	0.182020000000000
6	0.184313725490196	0.176651666666667
7	0.182957393483709	0.176634285714286
8	0.180351906158358	0.174892500000000
9	0.179111761835041	0.174100000000000
10	0.177728937728938	0.173551000000000
11	0.176791717985420	0.172699090909091
12	0.175914057254471	0.172270833333333
13	0.175219543799604	0.171813076923077
14	0.174600268122813	0.171547857142857
15	0.174075515170935	0.170988666666667
16	0.173610395852645	0.170941250000000
17	0.173202969668839	0.170482941176471
18	0.172839329566316	0.170408333333333
19	0.172514707721423	0.170206315789474
20	0.172222178512139	0.170242500000000
21	0.171957693718095	0.170073333333333

The Markov king densities differ from the Zeckendorf densities by less than $\frac{1}{100}$ of a king per tile. The Zeckendorf approximations were averaged over building 100 boards and we would expect the densities to be closer if the Zeckendorf approximations were averaged over a larger number of boards.

The king densities can be used to determine the expected number of kings for $m \times n$ boards for large n. For example in a 10×85 board,

$$\#\text{ kings} = \rho_{10} \times 10 \times 85 \approx 151 \text{ kings}.$$

In general, for an $m \times n$ board,

$$\#\text{ kings} = \rho_m \times m \times n.$$

For $m \geq 21$, Markov king densities could not be calculated in a reasonable time. The Zeckendorf density for a $100 \times n$ board is 0.168457000000000. The densities in the table and this Zeckendorff density on a $100 \times n$ board suggest that the densities may converge to some number less than 0.17 as $m \to \infty$.

Our main goal is to count the number of placements of kings on a $m \times n$ board, and this method using Markov chains gives us an expected number of kings on $m \times n$ boards for large n, but the relationship between the number of kings and the number of placements has not been found. Finding this may also give a relationship between king densities and the entropy constant η, which would give a method for approximating the entropy without using the eigenstructure of adjacency matrices as in previous king's configuration research.

6 $f_{m,n}^{\frac{1}{mn}}$ and its Series Expansion

6.1 Limit of $f_{1,n}^{\frac{1}{n}}$

Theorem 6.1. *The limit of the sequence $\{f_{1,n}^{\frac{1}{n}}\}_{n=0}^{\infty}$ exists*

Proof. We can show that this limit is the golden ratio, $\frac{1+\sqrt{5}}{2}$. Observe that

$$f_{1,n}^{\frac{1}{n}} = \frac{1+\sqrt{5}}{2}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n\right)^{\frac{1}{n}}$$

We can show that

$$\lim_{n\to\infty} f_{1,n}^{\frac{1}{n}} = \frac{1+\sqrt{5}}{2}$$

as the paranthesis tends to 1 when $n \to \infty$. $\square$

6.2 The Behavior of $f_{1,n}^{\frac{1}{n}}(x)$

Since

$$\lim_{n\to\infty} f_{1,n}^{\frac{1}{n}}(x) = \lim_{n\to\infty} \exp\left[\frac{1}{n}\log f_{1,n}(x)\right],$$

Let $h_{1,n}(x) = f_{1,n}^{\frac{1}{n}}(x) = \sum_{k\geq 0} h_k x^k$ and $f_{1,n}(x) = \sum_{k\geq 0} f_k x^k$ then we have

$$h'_{1,n}(x) f_{1,n}(x) = \frac{1}{n} h_{1,n}(x) f'_{1,n}(x).$$

By computing the coefficient of x^k in the left and right hand side of the previous equation, we obtain the following recurrence relation

$$h_k = \frac{1}{k}\sum_{i=1}^{k} h_{k-i} f_i \left(\frac{i}{n} - k + 1\right)$$

Note that $\lim_{n\to\infty} h_{1,n}(x) = 1 + x - x^2 + 2x^3 - 5x^4 + \cdots$.

We also have

$$h_0 = h_{1,n}(0) = \exp\left(\frac{1}{n}\log f_{1,n}(0)\right) = \exp(0) = 1.$$

Using the recurrence relation and the fact that

$$f_k = \binom{n-k+1}{k},$$

we have

$$\begin{aligned}
h_1 &= 1,\\
h_2 &= -1 + \frac{1}{n},\\
h_3 &= 2 - \frac{3}{n},\\
h_4 &= -5 + \frac{19}{2n} + \frac{1}{2n^2},\\
h_5 &= 14 - \frac{63}{2n} - \frac{7}{2n^2} \text{for n} \geq 4,\\
h_6 &= -42 + \frac{647}{6n} + \frac{108}{6n^2} + \frac{1}{6n^3} \text{for n} \geq 5,\\
h_7 &= 132 - \frac{2266}{6n} - \frac{495}{6n^2} - \frac{11}{6n^3} \text{for n} \geq 6
\end{aligned}$$

Computational data has shown that for $k \geq 2$,

$$\lim_{n\to\infty} h_k = (-1)^{k-1} C_{k-1}$$

where C_k is the k^{th} Catalan number.

Note: If we ignore the non-constant terms, for example, take $h_2 = -1$, then we will end up with $h_3 = \frac{8}{3}$. Thus the non-constant terms matter.

Theorem 6.2. *For $m \geq 2$,*

$$\lim_{n\to\infty} h_m = (-1)^{m-1} C_{m-1}$$

where C_m is the m-th Catalan number.

Proof. To prove this theorem, we need to consider $g(x) = \log f_{1,n}(x) = \sum_{i=0}^{\infty} g_i x^i$, the series expansion of the logarithm of $f_{1,n}(x)$ and look at its coefficients.

$$xg'(x) = x\frac{f'_{1,n}(x)}{f_{1,n}(x)}.$$

So

$$xg'(x)f_{1,n}(x) = xf'_{1,n}(x).$$

Note that $xg'(x) = \sum_{i=1}^{\infty} ig_i x^i$, thus

$$\left(\sum_{i=1}^{\infty} ig_i x^i\right)\left(\sum_{j=0}^{\infty} f_j x^j\right) = \left(\sum_{k=1}^{\infty} kf_k x^k\right),$$

and the coefficient of x^m is

$$\sum_{r=1}^{m} rg_r f_{m-r} = mf_m.$$

Then

$$g_0 = \log f(0) = 0.$$

Since $f_0 = f(0) = 1$, then

$$\begin{aligned}
g_1 &= \frac{1}{1}(n+0)\\
g_2 &= \frac{1}{2}(-3n+2)\\
g_3 &= \frac{1}{3}(10n-12)\\
g_4 &= \frac{1}{4}(-35n+58)\\
&\vdots
\end{aligned}$$

Thus we have the following theorem:

Theorem 6.3. *If we denote g_m as a function of n, i.e. $g_m(n)$, and $g_m(n) = a_m n + b_m$, then. for $m \geq 1$,*

$$g_m(n) = \frac{1}{m}(-1)^{m-1}\binom{2m-1}{m}n + (-1)^m\frac{2}{m}\left(4^{m-1} - \binom{2m-1}{m-1}\right).$$

So

$$\begin{aligned}
a_m &= \frac{1}{m}(-1)^{m-1}\binom{2m-1}{m}\\
b_m &= (-1)^m\frac{2}{m}\left(4^{m-1} - \binom{2m-1}{m-1}\right)
\end{aligned}$$

We will use the formula for a_m from this theorem to prove our previous theorem.

For our convenience, we denote

$$a(x) = \lim_{n\to\infty} \frac{1}{n} g(x) = \sum_{i=1}^{\infty} a_i x^i$$

Remember that our goal is to use the above assumption to prove that for $m \geq 1$,

$$lim_{n\to\infty} h_m = \frac{1}{m}(-1)^{m-1}\binom{2m-2}{m-1}.$$

Let $h(x) = lim_{n\to\infty} h_{1,n}(x) = \sum_{i\geq 0} \mu_i x^i$. Now, since

$$\begin{aligned} h(x) &= \lim_{n\to\infty} f^{\frac{1}{n}}(x) = \exp\left(\lim_{n\to\infty} \frac{1}{n}\log f(x)\right) = \exp\left(\lim_{n\to\infty} \frac{1}{n} g(x)\right) \\ &= \exp\left(a(x)\right), \end{aligned}$$

so

$$h'(x) = \exp\left(a(x)\right) a'(x) = h(x) a'(x)$$

So

$$xh'(x) = h(x)\left[xa'(x)\right].$$

Thus

$$\left(\sum_{i=1}^{\infty} i\mu_i x^i\right) = \left(\sum_{j=0}^{\infty} \mu_j x^j\right)\left(\sum_{k=1}^{\infty} k a_k x^k\right),$$

and the coefficient of x^m is

$$mh_m = \sum_{r=0}^{m-1} (m-r) a_{m-r} \mu_r.$$

We proceed by induction. Knowing

$$a_m = \frac{1}{m}(-1)^{m-1}\binom{2m-1}{m}$$

, then

$$h_m = \frac{1}{m}(-1)^{m-1}\binom{2m-2}{m-1}$$

.

Suppose this is true for $k \le m+1$, then for $k = m+2$,

$$\begin{aligned}(m+2)\mu_{m+2} &= \sum_{r=0}^{m+1}(m+2-r)a_{m+2-r}\mu_r \\ &= (m+2)a_{m+2}\mu_0 + \sum_{r=1}^{m+1}(m+2-r)a_{m+2-r}\mu_r \\ &= (m+2)a_{m+2} + \sum_{r=1}^{m+1}(m+2-r)a_{m+2-r}\mu_r\end{aligned}$$

We pull out the term when $r = 0$ because our formula does not satisfy the case when the index is zero. Next, change the index in the sum, i.e. let $k = r - 1$, then $r = k + 1$ and

$$\begin{aligned}(m+2)\mu_{m+2} - (m+2)a_{m+2} &= \sum_{k=0}^{m}(m+1-k)a_{m+1-k}\mu_{k+1} \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{2m-2k+1}{m-k+1}\frac{1}{k+1}(-1)^k\binom{2k}{k} \\ &= (-1)^m\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k+1}{m-k+1}\end{aligned}$$

The following lemma is trivial.

Lemma 6.4.

$$\binom{2m-2k+1}{m-k+1} = \left(2 - \frac{1}{m-k+1}\right)\binom{2m-2k}{m-k}.$$

Hence, we have

$$\begin{aligned}&(-1)^m\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k+1}{m-k+1} \\ &= (-1)^m 2\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k} - (-1)^m\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\frac{1}{m-k+1}\binom{2m-2k}{m-k}\end{aligned}$$

Lemma 6.5.

$$2\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k} = (m+2)\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\frac{1}{m-k+1}\binom{2m-2k}{m-k}.$$

Proof.

$$2\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k} = \sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k} + \sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k}$$
$$= \sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\binom{2m-2k}{m-k} + \sum_{t=0}^{m}\frac{1}{m-t+1}\binom{2m-2t}{m-t}\binom{2t}{t},$$

for $t = m-k$.

Then we have

$$= \sum_{k=0}^{m}\binom{2k}{k}\binom{2m-2k}{m-k}\left[\frac{1}{k+1}+\frac{1}{m-k+1}\right]$$
$$= (m+2)\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\frac{1}{m-k+1}\binom{2m-2k}{m-k}$$

□

Lemma 6.6.

$$\sum_{k=0}^{m}\frac{1}{k+1}\binom{2k}{k}\frac{1}{m-k+1}\binom{2m-2k}{m-k} = \frac{1}{m+2}\binom{2m+2}{m+1}.$$

Proof. The formula is equivalent to $\sum_{k=0}^{m} C_k C_{m-k} = C_{m+1}$, where C_k is the k^{th} Catalan number. This formula is called Segner's recurrence relation and it's proved using the paranthesis counting method, in [1], pgs 43-44.

Therefore,

$$(m+2)\mu_{m+2} - (m+2)a_{m+2} = (-1)^m(m+1)\frac{1}{m+2}\binom{2m+2}{m+1}$$

From here, by simple computation, we get

$$(m+2)\mu_{m+2} = (-1)^{m+1}\binom{2m+2}{m+1},$$

□

So $h(x) = \lim_{n\to\infty} h_{1,n}(x) = \sum_{k\geq 0}(-1)^{k-1}C_{k-1}x^k$

□

6.3 Binary Strings, $f_{1,n}$ and Formula of a_m

Here we will prove the formula for a_m in the previous theorem,

$$g_m(n) = \frac{1}{m}(-1)^{m-1}\binom{2m-1}{m}n + (-1)^m \frac{2}{m}\left(4^{m-1} - \binom{2m-1}{m-1}\right).$$

Notation:

$$g_m(n) = a_m n + b_m$$

We consider the regular language S, denote by S^*, the set of all words formed by concatenating a finite number of elements of S. A regular language has the property that each word is obtained uniquely.

Example 6.7. *Suppose $S = \{0, 1\}$, then*

$$S^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \cdots\}$$

where ϵ denote the empty string.

Example 6.8. *The set $\{0, 1, 01\}$ is not a part of this regular language, since the string 01 is obtained as 0, 1 and also as 01, so not obtained uniquely.*

Let $S = \{0, 1\}$ and denote $S^* =$ all binary sequences and write

$$\{0\}^*\{1\{1\}^*0\{0\}^*\}^*\{1\}^*$$

as

$$0^*(11^*00^*)^*1^*.$$

We now focus on the set of binary strings without consecutive 1's.

$$0^*(100^*)^*(\epsilon \cup 1)$$

$$= \{\epsilon, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \cdots\}.$$

We want to compute the generating function for such strings, with x marking length.

$$1 + 2x + 3x^2 + 5x^3 + \cdots$$

Lemma 6.9. 0^* *has generating function*

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

This is because

$$0^* = \{\epsilon, 0, 00, 000, 0000, \cdots\}.$$

Lemma 6.10. 100^* *has generating function*

$$\frac{x^2}{1-x}.$$

This is because
$$100^* = \{10, 100, 1000, 10000, \cdots\}.$$

Lemma 6.11. $(100^*)^*$ *has generating function*

$$\frac{1-x}{1-x-x^2}.$$

This comes from

$$1+\left(\frac{x^2}{1-x}\right)+\left(\frac{x^2}{1-x}\right)^2+\left(\frac{x^2}{1-x}\right)^3+\cdots=\frac{1}{1-\left(\frac{x^2}{1-x}\right)}.$$

Lemma 6.12. $\epsilon \cup 1$ *has generating function*

$$1+x.$$

This is because
$$\{\epsilon \cup 1\} = \{\epsilon, 1\}.$$

Therefore,

Theorem 6.13. $0^*(100^*)^*(\epsilon \cup 1)$ *has generating function*

$$\frac{1+x}{1-x-x^2}.$$

This is because
$$\frac{1}{1-x}\cdot\frac{1-x}{1-x-x^2}\cdot(1+x)=\frac{1+x}{1-x-x^2}.$$

Suppose
$$f(x)=\frac{1+x}{1-x-x^2}=\sum_{k=0}^{\infty} f_k x^k,$$

then
$$(1-x-x^2)f(x) = 1+x.$$

Therefore,

$$[x^n]\left[(1-x-x^2)f(x)\right] = [x^n]\,(1+x) = 1x^0+1x^1+0x^2+0x^3+0x^4+\ldots$$

Thus,
$$[x^0] = 1$$

$$[x^1] = 1$$

For $n \geq 2$

$$[x^n] = 0$$

On the other hand, for $n \geq 2$,

$$[x^n]\left(f(x) - xf(x) - x^2 f(x)\right) = f_n - f_{n-1} - f_{n-2}.$$

Hence for $n \geq 2$,

$$f_n = f_{n-1} + f_{n-2}.$$

The set

$$0^*(100^*)^*(\epsilon \cup 1)$$

of binary strings without adjacent 1's corresponds to the number of configurations of non attacking kings on a board. Consider the generating function

$$f(x, y) = \sum_{n,k \geq 0} Bf_{n,k} x^n y^k$$

where $Bf_{n,k} = \#$ binary strings of length n with exactly k 1's. Thus each $Bf_{n,k}$ is finite.

In this generating function x counts the length of the string and y counts the number of 1's in the string.

Then each 0 contributes x and each 1 contributes xy.

Lemma 6.14. 0^* *has generating function*

$$\frac{1}{1-x}.$$

This is true because $0^* = \{\epsilon, 0, 00, 000, \ldots\}$ so the lengths of the strings are $0, 1, 2, 3, 4, 5, \ldots$ counted by $1 + x + x^2 + x^3 + \ldots = \sum_{k \geq 0} x^k = \frac{1}{1-x}$, the geometric series.

Lemma 6.15. 100^* *has generating function*

$$\frac{x^2 y}{1-x}.$$

We have the same generating function from the previous lemma, $\frac{1}{1-x}$, but have an additional 1 and 0 at the beginning of the string which contribute xy and x respectively to form the new generating function.

Lemma 6.16. $(100^*)^k$ *has generating function*

$$\left(\frac{x^2 y}{1-x}\right)^k.$$

Therefore,

Lemma 6.17. $(100^*)^*$ *has generating function*

$$\frac{1}{1-\frac{x^2y}{1-x}}.$$

The generating function for 0^* is $\frac{1}{1-x}$, and the generating function for 0 is x. Since we have $(100^*)^*$, we replace the generating function for 0 (which is x) in $\frac{1}{1-x}$ with the generating function for 100^*, which is is $\frac{x^2y}{1-x}$.

Lemma 6.18. $\epsilon \cup 1$ *has generating function*

$$1+xy.$$

Therefore,

Theorem 6.19. $0^*(100^*)^*(\epsilon \cup 1)$ *has generating function*

$$\frac{1+xy}{1-x-x^2y}.$$

We want to solve

$$(1-x-x^2y) = (1-\alpha(y)x)(1-\beta(y)x),$$

then

$$f_{1,n}(y) = A(y)\alpha(y)^n + B(y)\beta(y)^n,$$

where

$$\frac{A(y)}{1-\alpha(y)x} + \frac{B(y)}{1-\beta(y)x} = \frac{1+xy}{(1-\alpha(y)x)(1-\beta(y)x)}.$$

and to find $A(y)$ and $B(y)$.

We get that $\alpha(y)\beta(y) = -y$ and $\alpha(y)+\beta(y) = 1$. Thus, $\alpha(y)^2 - \alpha(y) - y = 0$. Solving for the two roots, choose

$$\alpha(y) = \frac{1+\sqrt{1+4y}}{2} \text{ and } \beta(y) = \frac{1-\sqrt{1+4y}}{2}.$$

We get that

$$A(y)(1-\beta(y)x) + B(y)(1-\alpha(y)x) = 1+xy$$

Setting x to be equal to $\frac{1}{\alpha(y)}$ and $\frac{1}{\beta(y)}$, we get the formulas for $A(y)$ and $B(y)$:

$$A(y) = \frac{1+4y+(1+2y)\sqrt{1+4y}}{8y+2}$$

$$B(y) = \frac{1+4y-(1+2y)\sqrt{1+4y}}{8y+2}$$

Now,

$$f_{1,n}(y) = \alpha(y)^n A(y) \left(1 + \frac{B(y)}{A(y)} \left(\frac{\beta(y)}{\alpha(y)}\right)^n\right)$$

and

$$f_{1,n}^{\frac{1}{n}}(y) = \alpha(y) A(y)^{\frac{1}{n}} \left(1 + \frac{B(y)}{A(y)} \left(\frac{\beta(y)}{\alpha(y)}\right)^n\right)^{\frac{1}{n}}$$

We observe that

$$\left(1 + \frac{B(y)}{A(y)} \left(\frac{\beta(y)}{\alpha(y)}\right)^n\right)^{\frac{1}{n}} = 1 + O(y^n)$$

meaning that, as a power series in y, the coefficients of y^k are zero for $1 \leq k \leq n-1$

Thus,

$$\log f_{1,n}(y) = n \log(\alpha(y)) + \log A(y) + O(y^n) = n(\sum_{m \geq 0} a_m y^m) + \sum_{m \geq 0} b_m y^m$$

Observation: Due to the $O(y^n)$, the formulas for the coefficients a_m and b_m might not hold for $m \geq n$, since we have not included the contribution from this last term. However, this is not important since we're taking limit as n tends to infinity to prove the conjecture for h_m.

We want that the expansion in series of $\log(\alpha(y))$ to be equal to $\sum\limits_{m \geq 0} a_m y^m$.

We know that $\alpha(y) = \frac{1+\sqrt{1+4y}}{2} = \sum\limits_{k \geq 0} (-1)^{k-1} C_{k-1} y^k$, defining by convention $C_{-1} = 1$ and C_k the kth Catalan number in general.

By the same method through which we obtained the recurrence relation for g_m, by differentiating both sides of the equality and isolating the leading coefficient, we obtain

$$(m+1) a_{m+1} = (m+1)(-1)^m C_m - \sum_{k=1}^{m} k a_k (-1)^{m-k} C_{m-k}$$

Wanting $a_k = \frac{1}{k}(-1)^{k-1}\binom{2k-1}{k}$, we use this formula in the recurrence relation and prove that the corresponding generating functions are the same for both sides. Thus, we obtain (using **Proposition 5.26**, which states that $\sum_{r=0}^{m} 4^r \binom{2m-2r}{m-r} = (2m+1)\binom{2m}{m}$)

$$(m+1)(-1)^m \frac{1}{m+1}\binom{2m+1}{m} =$$
$$= (m+1)(-1)^m \frac{1}{m+1}\binom{2m}{m} - \sum_{k=1}^{m} k \frac{1}{k} (-1)^{k-1} \binom{2k-1}{k} (-1)^{m-k} \frac{1}{m-k+1}\binom{2m-2k}{m-k}$$

So

$$\binom{2m+1}{m} = \binom{2m}{m} + \sum_{k=1}^{m} \frac{1}{m-k+1} \binom{2k-1}{k} \binom{2m-2k}{m-k}$$
$$\binom{2m+1}{m} = \binom{2m}{m} + \sum_{k=1}^{m} \frac{1}{m-k+1} \binom{2k-1}{k} \binom{2m-2k}{m-k}$$
$$\binom{2m+1}{m} = \binom{2m}{m} + \frac{1}{2} \sum_{k=0}^{m} \frac{1}{m-k+1} \binom{2k}{k} \binom{2m-2k}{m-k} - \frac{1}{2(m+1)} \binom{2m}{m} \binom{0}{0}$$
$$\binom{2m+1}{m} = \binom{2m}{m} \left(1 - \frac{1}{2(m+1)}\right) + \frac{1}{4} \binom{2m+2}{m+1}$$
$$\binom{2m+1}{m} = \binom{2m}{m} \frac{2m+1}{2(m+1)} + \frac{1}{2} \binom{2m+1}{m}$$

Thus, we expand the binomial coefficients.

$$\frac{(2m+1)2m\cdots(m+2)}{m!} = \frac{2m+1}{2(m+1)} \frac{2m\cdots(m+2(m+1))}{m!} + \frac{1}{2} \frac{(2m+1)2m\cdots(m+2)}{m!}$$
$$\Rightarrow 2m+1 = \frac{2m+1}{2} + \frac{2m+1}{2}$$

Hence, the expansion in series of $\log \alpha(y)$ is equal to $\sum\limits_{k=0} (-1)^{k-1} \binom{2k-1}{k} y^k$. Since it is also equal to $\sum\limits_{k=0} a_k y^k$, this proves the formula for a_k.

6.4 Formula for b_m

Corollary 6.20. *For $m \geq 1$,*

$$b_m = g_m(0) = (-1)^m \frac{1}{2m} 4^m - (-1)^m \frac{1}{m} \binom{2m}{m}$$

Proof. Note that

$$(-1)^m \frac{2}{m} \left(4^{m-1} - \binom{2m-1}{m-1}\right) = (-1)^m \frac{2}{m} 4^{m-1} - (-1)^m \frac{2}{m} \binom{2m-1}{m-1}$$
$$= (-1)^m \frac{1}{2m} 4^m - (-1)^m \frac{1}{m} \binom{2m}{m},$$

We proceed by induction over m. Suppose this holds for all $r < m+1$ and prove it holds for

$m+1$, using $f_0 = 1$ and the recurrence relation

$$\begin{aligned}(m+1)g_{m+1} &= (m+1)f_{m+1} - \sum_{r=1}^{m} rg_r f_{m+1-r} \\ &= (m+1)f_{m+1} - \sum_{r=1}^{m-1} rg_r f_{m+1-r} - mg_m f_1.\end{aligned}$$

Let $n = 0$ on both sides, since $f_1(0) = 0$, we obtain

$$(m+1)g_{m+1}(0) = (m+1)f_{m+1}(0) - \sum_{r=1}^{m-1} rg_r(0) f_{m+1-r}(0).$$

Notation:

$$\begin{gathered}(m+1)g_{m+1}(0) = A_1 - A_2 \\ A_1 = (m+1)f_{m+1}(0) \\ A_2 = \sum_{r=1}^{m-1} rg_r(0) f_{m+1-r}(0)\end{gathered}$$

We will compute A_2.

Note that

$$\begin{aligned}A_2 &= \sum_{r=1}^{m-1} rg_r(0) f_{m+1-r}(0) \\ &= \sum_{r=1}^{m-1} r\left[(-1)^r \frac{1}{2r}4^r - (-1)^r \frac{1}{r}\binom{2r}{r}\right](-1)^{m+1-r}\binom{2m-2r}{m+1-r} \\ &= \sum_{r=1}^{m-1} r(-1)^r \frac{1}{2r}4^r (-1)^{m+1-r}\binom{2m-2r}{m+1-r} - \sum_{r=1}^{m-1} r(-1)^r \frac{1}{r}\binom{2r}{r}(-1)^{m+1-r}\binom{2m-2r}{m+1-r} \\ &= B_1 - B_2\end{aligned}$$

Simplify the first term, we have

$$\begin{aligned}B_1 &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \binom{2m-2r}{m+1-r} \\ &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \frac{m-r}{m+1-r}\binom{2m-2r}{m-r} \\ &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \left(1 - \frac{1}{m-r+1}\right)\binom{2m-2r}{m-r} \\ &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \binom{2m-2r}{m-r} - \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r}.\end{aligned}$$

Note that, due to **Proposition 6.23**, we have

$$\begin{aligned}\sum_{r=1}^{m-1} 4^r \binom{2m-2r}{m-r} &= \sum_{r=0}^{m} 4^r \binom{2m-2r}{m-r} - \binom{2m}{m} - 4^m \\ &= (2m+1)\binom{2m}{m} - \binom{2m}{m} - 4^m \\ &= 2m\binom{2m}{m} - 4^m,\end{aligned}$$

and, due to **Proposition 6.24**, we have

$$\begin{aligned}\sum_{r=1}^{m-1} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r} &= \sum_{r=0}^{m} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r} - \frac{1}{m+1}\binom{2m}{m} - 4^m \\ &= \frac{1}{2}\cdot 4^{m+1} - \frac{1}{2}\binom{2m+2}{m+1} - \frac{1}{m+1}\binom{2m}{m} - 4^m \\ &= 2\cdot 4^m - \frac{2m+1}{m+1}\binom{2m}{m} - \frac{1}{m+1}\binom{2m}{m} - 4^m \\ &= 4^m - 2\binom{2m}{m}.\end{aligned}$$

Thus

$$\begin{aligned}B_1 &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \binom{2m-2r}{m+1-r} \\ &= \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \binom{2m-2r}{m-r} - \frac{1}{2}(-1)^{m+1}\sum_{r=1}^{m-1} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r} \\ &= \frac{1}{2}(-1)^{m+1}\left[2m\binom{2m}{m} - 4^m - 4^m + 2\binom{2m}{m}\right] \\ &= (-1)^{m+1}\left[(m+1)\binom{2m}{m} - 4^m\right].\end{aligned}$$

So

$$B_1 = (-1)^{m+1}\left[(m+1)\binom{2m}{m} - 4^m\right]$$

Next, compute B_2 in

$$\sum_{r=1}^{m-1} r g_r(0) f_{m+1-r}(0) = B_1 - B_2,$$

then we have

$$
\begin{aligned}
B_2 &= (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\binom{2m-2r}{m+1-r}\\
&= (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\frac{m-r}{m+1-r}\binom{2m-2r}{m-r}\\
&= (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\left(1-\frac{1}{m-r+1}\right)\binom{2m-2r}{m-r}\\
&= (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\binom{2m-2r}{m-r} - (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r}.
\end{aligned}
$$

Note that, due to **Proposition 6.25**, we have

$$
\begin{aligned}
\sum_{r=1}^{m-1}\binom{2r}{r}\binom{2m-2r}{m-r} &= \sum_{r=0}^{m}\binom{2r}{r}\binom{2m-2r}{m-r} - 2\binom{2m}{m}\\
&= 4^m - 2\binom{2m}{m},
\end{aligned}
$$

and, due to **Proposition 6.26**, we have

$$
\begin{aligned}
&\sum_{r=1}^{m-1}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r}\\
&= \sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r} - \frac{1}{m+1}\binom{2m}{m} - \binom{2m}{m}\\
&= \frac{1}{2}\binom{2m+2}{m+1} - \frac{m+2}{m+1}\binom{2m}{m}\\
&= \frac{2m+1}{m+1}\binom{2m}{m} - \frac{m+2}{m+1}\binom{2m}{m}\\
&= \frac{m-1}{m+1}\binom{2m}{m}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_2 &= (-1)^{m+1}\sum_{r=1}^{m-1}\binom{2r}{r}\binom{2m-2r}{m+1-r}\\
&= (-1)^{m+1}\left[4^m - 2\binom{2m}{m}\right] - (-1)^{m+1}\frac{m-1}{m+1}\binom{2m}{m}\\
&= (-1)^{m+1}\left[4^m - 2\binom{2m}{m} - \frac{m-1}{m+1}\binom{2m}{m}\right]\\
&= (-1)^{m+1}\left[4^m - \frac{3m+1}{m+1}\binom{2m}{m}\right].
\end{aligned}
$$

So

$$B_2 = (-1)^{m+1} \left[4^m - \frac{3m+1}{m+1} \binom{2m}{m} \right]$$

Hence

$$\begin{aligned}
A_2 &= B_1 - B_2 \\
&= (-1)^{m+1} \left[(m+1) \binom{2m}{m} - 4^m \right] - (-1)^{m+1} \left[4^m - \frac{3m+1}{m+1} \binom{2m}{m} \right] \\
&= (-1)^{m+1} \left[\frac{m^2+5m+2}{m+1} \binom{2m}{m} - 2 \cdot 4^m \right].
\end{aligned}$$

$$A_2 = (-1)^{m+1} \left[\frac{m^2+5m+2}{m+1} \binom{2m}{m} - 2 \cdot 4^m \right].$$

Now,

$$\begin{aligned}
A_1 = (m+1) f_{m+1}(0) &= (m+1)(-1)^{m+1} \binom{2m}{m+1} \\
&= m(-1)^{m+1} \binom{2m}{m}.
\end{aligned}$$

So

$$A_1 = m(-1)^{m+1} \binom{2m}{m}.$$

Therefore,

$$\begin{aligned}
(m+1) g_{m+1}(0) &= A_1 - A_2 \\
&= (-1)^{m+1} m \binom{2m}{m} - (-1)^{m+1} \left[\frac{m^2+5m+2}{m+1} \binom{2m}{m} - 2 \cdot 4^m \right] \\
&= (-1)^{m+1} \left[m \binom{2m}{m} - \frac{m^2+5m+2}{m+1} \binom{2m}{m} + 2 \cdot 4^m \right] \\
&= (-1)^{m+1} \left[m \binom{2m}{m} - \frac{m^2+5m+2}{m+1} \binom{2m}{m} + 2 \cdot 4^m \right] \\
&= (-1)^{m+1} \left[\frac{-4m-2}{m+1} \binom{2m}{m} + 2 \cdot 4^m \right] \\
&= (-1)^{m+1} \frac{1}{2} 4^{m+1} - (-1)^{m+1} \frac{4m+2}{m+1} \binom{2m}{m} \\
&= (-1)^{m+1} \frac{1}{2} 4^{m+1} - (-1)^{m+1} \binom{2m+2}{m+1}.
\end{aligned}$$

$\square$

6.5 Prerequisites for b_m

We need some lemmas to prove the following propositions.

Lemma 6.21.

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}.$$

Proof. Note that

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k.$$

Then

$$\frac{1-\sqrt{1-4x}}{2} = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^{k+1}.$$

Thus

$$\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k.$$

□

Lemma 6.22.

$$\binom{-\frac{3}{2}}{k} = \left(\frac{-1}{4}\right)^k (2k+1)\binom{2k}{k}.$$

Proof.

$$\begin{aligned}
\binom{-\frac{3}{2}}{k} &= \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\cdots\left(-\frac{3}{2}-k+1\right)}{k!} \\
&= \left(-\frac{1}{2}\right)^k \frac{3\cdot 5\cdots(2k+1)}{k!} \\
&= \left(-\frac{1}{2}\right)^k \frac{2\cdot 4\cdots(2k)}{2^k k!} \frac{3\cdot 5\cdots(2k+1)}{k!} \\
&= \left(-\frac{1}{4}\right)^k (2k+1)\binom{2k}{k}.
\end{aligned}$$

□

Proposition 6.23.

$$\sum_{r=0}^{m} 4^r \binom{2m-2r}{m-r} = (2m+1)\binom{2m}{m}.$$

Proof.

$$\begin{aligned}\sum_{r=0}^{m} 4^r \binom{2m-2r}{m-r} &= [x^m]\left(\sum_{i=0}^{\infty} 4^i x^i\right)\left(\sum_{j=0}^{\infty}\binom{2j}{j}x^j\right)\\ &= [x^m]\left(\frac{1}{1-4x}\right)\left(\frac{1}{\sqrt{1-4x}}\right)\\ &= [x^m]\left((1-4x)^{-\frac{3}{2}}\right)\\ &= \binom{-\frac{3}{2}}{m}(-4)^m\\ &= \left(-\frac{1}{4}\right)^m (2m+1)\binom{2m}{m}(-4)^m\\ &= (2m+1)\binom{2m}{m}.\end{aligned}$$

□

Proposition 6.24.

$$\sum_{r=0}^{m} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r} = \frac{1}{2}4^{m+1} - \frac{1}{2}\binom{2m+2}{m+1}.$$

Proof.

$$\begin{aligned}\sum_{r=0}^{m} 4^r \frac{1}{m-r+1}\binom{2m-2r}{m-r} &= [x^m]\left(\sum_{i=0}^{\infty} 4^i x^i\right)\left(\sum_{j=0}^{\infty}\frac{1}{j+1}\binom{2j}{j}x^j\right)\\ &= [x^m]\left(\frac{1}{1-4x}\right)\left(\frac{1-\sqrt{1-4x}}{2x}\right)\\ &= [x^{m+1}]\left(\frac{1}{1-4x}\right)\left(\frac{1-\sqrt{1-4x}}{2}\right)\\ &= \frac{1}{2}[x^{m+1}]\left(\frac{1}{1-4x}\right) - \frac{1}{2}[x^{m+1}]\left(\frac{1}{\sqrt{1-4x}}\right)\\ &= \frac{1}{2}4^{m+1} - \frac{1}{2}\binom{2m+2}{m+1}.\end{aligned}$$

□

Proposition 6.25.

$$\sum_{r=0}^{m}\binom{2r}{r}\binom{2m-2r}{m-r} = 4^m.$$

Proof.

$$\begin{aligned}\sum_{r=0}^{m}\binom{2r}{r}\binom{2m-2r}{m-r} &= [x^m]\left(\sum_{i=0}^{\infty}\binom{2i}{i}x^i\right)\left(\sum_{j=0}^{\infty}\binom{2j}{j}x^j\right)\\ &= [x^m]\left(\frac{1}{\sqrt{1-4x}}\right)\left(\frac{1}{\sqrt{1-4x}}\right)\\ &= [x^m]\left(\frac{1}{1-4x}\right)\\ &= 4^m.\end{aligned}$$

□

Proposition 6.26.

$$\sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r} = \frac{1}{2}\binom{2m+2}{m+1}.$$

Proof. Consider

$$\begin{aligned}&2\sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r}\\ &= \sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r} + \sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r}\\ &= \sum_{r=0}^{m}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r} + \sum_{t=0}^{m}\binom{2m-2t}{m-t}\frac{1}{t+1}\binom{2t}{t}\\ &= \sum_{r=0}^{m}\binom{2r}{r}\binom{2m-2r}{m-r}\left[\frac{1}{m-r+1}+\frac{1}{r+1}\right]\\ &= (m+2)\sum_{r=0}^{m}\frac{1}{r+1}\binom{2r}{r}\frac{1}{m-r+1}\binom{2m-2r}{m-r}\\ &= \binom{2m+2}{m+1}.\end{aligned}$$

□

Thus we know that $\lim_{n\to\infty} f_{1,n}^{\frac{1}{n}}(x) = \lim_{n\to\infty} h_{1,n}(x) = \sum_{k\geq 1}(-1)^{k-1}C_{k-1}x^k = \frac{1+\sqrt{1+4x}}{2}$.

6.6 Limit of $h_{2,n}(x)$ and beyond

Lemma 6.27. $f_{2,n}(x) = f_{1,n}(2x)$

The above lemma gives us $\lim_{n\to\infty} f_{2,n}^{\frac{1}{2n}}(x) = \lim_{n\to\infty} f_{1,n}^{\frac{1}{2n}}(2x) = \left(\frac{1+\sqrt{1+8x}}{2}\right)^{\frac{1}{2}}$.
What about $f_{3,n}$ and beyond?

7 Polynomial Coefficient of y^k in $f_{m,n}$

In this section we will provide three methods to compute the coefficient of x^k in $f_{m,n}(x)$, using Lagrange interpolation, counting strategies and recurrence relations.

7.1 Lagrange Interpolation

Looking at the generating function of $f_{m,n}$, we observe that is symmetric in m and $n \Rightarrow f_{m,n}(x) = f_{n,m}(x)$.

Conjecture 7.1. *For $k \geq 0$, we have $[x^k](f_{m,n}(x)) = P_k(m,n)$, where $P_k \in Q[x]$ is a symmetric polynomial of degree k in both m and n.*

However, this polynomial works only for m and n large enough, so far, to be sure, w are using the lower bound $2k-1$ until we find a better one.

Method: For a fixed k, we can find the polynomial using the Lagrange interpolation in two different ways. Write $[x^k]f_{m,n}(x) = \sum\limits_{i=0}^{k} a_i(n)m^i$. Now we range n from $2k-1$ to $3k-1$ and for each of them we range m in the same interval. Thus, for a fixed $n = j$, we have $k+1$ values of m and thus $k+1$ values of $[x^k]f_{m,n}(x)$. Interpolating over m, we obtain the values $a_1(j)$, $a_2(j) \cdots a_k(j)$. Varying n, we obtain these values for $j \in \{2k-1, \cdots 3k-1\}$. Now, we interpolate for each of the polynomials $a_i(j)$ since we have $k+1$ values for each and we obtain $a_i(n), \forall i$. Thus we obtain $P_k(m,n)$. We observed that the polynomials obtained were valid even for values of m and n less than $2k-1$.

Corollary 7.2. *For $m, n \geq 1$,*

$$P_2(m,n) = \frac{1}{2}m^2n^2 - \frac{9}{2}mn + 3m + 3n - 2$$

Corollary 7.3. *For $m, n \geq 2$,*

$$P_3(m,n) = \frac{1}{6}m^3n^3 - \frac{9}{2}m^2n^2 + 3m^2n + 3n^2m + \frac{91}{3}mn - 38m - 38n + 44$$

.

For $n = 1$, if m starts from 1, we get the null polynomial. If m starts from 2 we get $\frac{1}{6}m^3 - \frac{3}{2}m^2 + \frac{13}{3}m - 4$, which differs from the general polynomial by $\frac{n}{3} + 10$. Lastly, for $n = 2$ and $m = 1$, we get the null polynomial.

Corollary 7.4. *For $m, n \geq 3$*

$$P_4(m,n) = \frac{1}{24}m^4n^4 - \frac{9}{4}m^3n^3 + \frac{3}{2}m^3n^2 + \frac{3}{2}n^3m^2 + \frac{995}{24}m^2n^2 - \frac{103}{2}m^2n - \frac{103}{2}n^2m + \frac{9}{2}m^2 + \frac{9}{2}n^2 - \frac{885}{4}mn + \frac{895}{2}m$$

However, for $k = 5$, we do not obtain a symmetric polynomial if we start with m and n from 4, keeping the linear increment of the lower bounds.

We can use these polynomials to compute the coefficients of $h_{m,n}(x) = f_{m,n}^{\frac{1}{mn}}(x)$. We have proved that for $m = 1$, the coefficients tend to $(-1)^{k-1}C_{k-1}$, but now we can compute the first five coefficients in the general case.

Denote $[x^k](h_{m,n}(x)) = h_k$. Again, by differentiating the equation and multiplying by x, we obtain a recurrence relation from which we get:

$$\begin{aligned} h_0 &= h_1 = 1 \\ h_2 &= -4 + \frac{3}{m} + \frac{3}{n} - \frac{2}{mn} \\ h_3 &= 28 - \frac{35}{m} - \frac{35}{n} + \frac{42}{mn} \\ h_4 &= -243 + \frac{807}{2m} + \frac{807}{2n} - \frac{654}{mn} + \frac{9}{2m^2} + \frac{9}{2n^2} - \frac{6}{mn^2} - \frac{6}{m^2n} + \frac{2}{m^2n^2} \end{aligned}$$

We have not yet found a pattern for the constant terms of h_i. As we observed, we need a great amount of data to interpolate, $(k+1)^2$ powers of the adjaceny matrix for each polynomial coefficient of x^k. Thus, we try a combinatorial, alternative method which is not yet generalized for $k \geq 5$.

7.2 Alternative Combinatorial Method

There are also combinatorial methods to compute the first polynomial coefficients of $f_{m,n}(x)$, by counting the arrangement of k kings on a board for k sufficiently small ($k \in \{1, 2, 3, 4\}$).

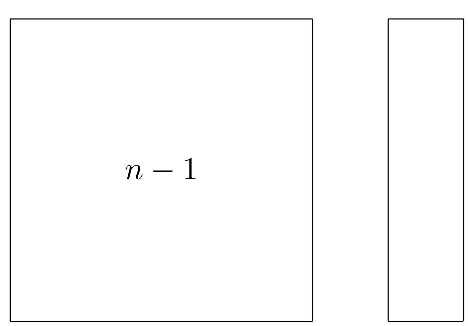
Here we will present the proof for $[x^3]f_{m,n}(x)$. For convenience, write $[x^k]f_{m,n}(x)$ as $[x^k]f_{m,n}$

Theorem 7.5. *For $n \geq 2$,*

$$[x^3]f_{4,n} = \frac{1}{3}(32n^3 - 180n^2 + 394n - 324).$$

Assume that $m \geq 5$ and consider how many kings there are on the last column.

If there are 0 kings on the last column, shown as below,



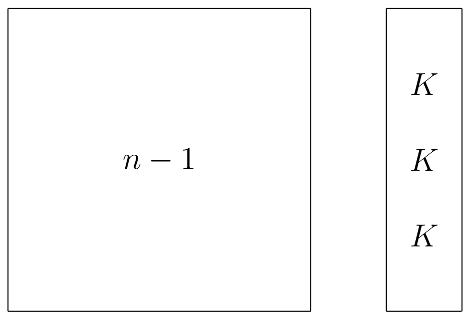
then we must have 3 kings on the $m \times (n-1)$ chessboard. Thus $n-1 \geq 1$, so $n \geq 2$ and the number of placements is

$$[x^3]f_{m,n-1} \times 1.$$

Note that

- if $3 \leq m \leq 4$, then $n-1 \geq 3$, so $n \geq 4$, and
- if $1 \leq m \leq 2$, then $n-1 \geq 5$, so $n \geq 6$.

If there are 3 kings on the last column, shown as below,

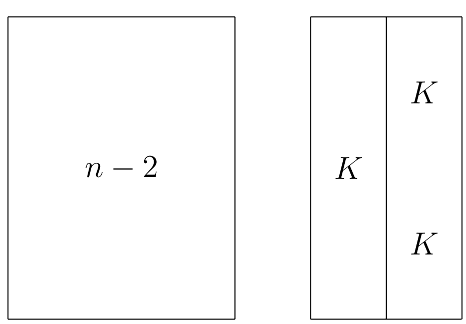


then we must have 0 kings on the $m \times (n-1)$ chessboard. Thus $n-1 \geq 0$, so $n \geq 1$ and the number of placements is

$$\begin{aligned}
&1 \cdot [x^3]f_{m,1} \\
&= [x^3]f_{1,m} \\
&= \binom{m-3+1}{3} \\
&= \frac{1}{6}(m^3 - 9m^2 + 26m - 24)
\end{aligned}$$

Note that this case will not occur when $m \leq 4$.

If there are 2 kings on the last column, then we have 2 possible situations. The first one is that the remaining king lies on the column next to the last column, shown as below,

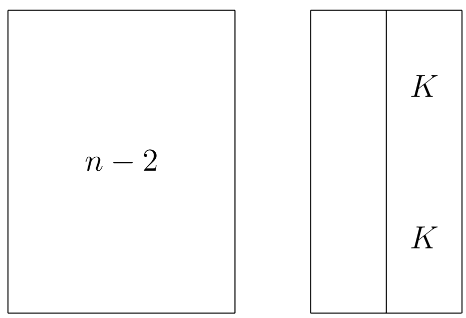


then we must have 0 kings on the $m \times (n-2)$ chessboard. Thus $n-2 \geq 0$, so $n \geq 2$ and the number of placements is

$$\begin{aligned}
1 \cdot \frac{1}{2}\left([x^3]f_{m,2} - 2 \cdot [x^3]f_{m,1}\right) &= \frac{1}{2}[x^3]f_{2,m} - [x^3]f_{1,m} \\
&= \frac{1}{2} \cdot 2^3 \cdot \binom{m-3+1}{3} - \binom{m-3+1}{3} \\
&= 3 \cdot \binom{m-2}{3} \\
&= \frac{1}{2}(m^3 - 9m^2 + 26m - 24)
\end{aligned}$$

Note that this case will not occur when $m \leq 4$.

The second case is that the remaining king lies on the column not adjacent to the last column, shown as below,

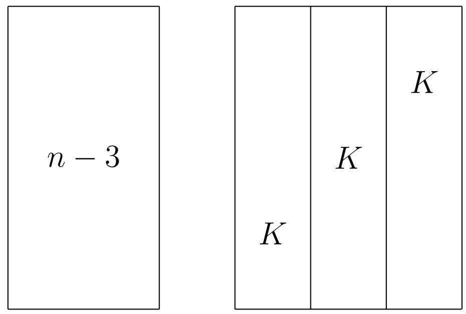


then we must have 1 kings on the $m \times (n-2)$ chessboard. Thus $n-2 \geq 1$, so $n \geq 3$ and the number of placements is

$$\begin{aligned}
[x^1]f_{m,n-2} \cdot [x^2]f_{m,1} &= [x^1]f_{m,n-2} \cdot [x^2]f_{1,m} \\
&= m(n-2) \cdot \binom{m-2+1}{2} \\
&= \frac{1}{2}(m^3 - 3m^2 + 2m)(n-2) \\
&= \frac{1}{2}(m^3 - 3m^2 + 2m)n - (m^3 - 3m^2 + 2m)
\end{aligned}$$

Note that this case will not occur when $m \leq 2$.

If there is only 1 king on the last column, then we have 4 possible situations. The first one is shown as below,

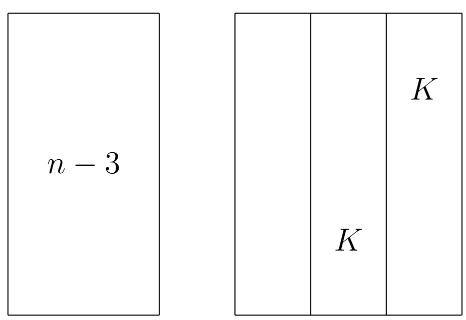


then we must have 0 kings on the $m \times (n-3)$ chessboard. Thus $n-3 \geq 0$, so $n \geq 3$ and the number of placements is

$$\begin{aligned}
& [x^3]f_{m,3} - 2 \cdot \big([x^3]f_{m,2} - 2 \cdot [x^3]f_{m,1}\big) - 2 \cdot \big([x^2]f_{m,1} \cdot [x^1]f_{m,1}\big) - 3 \cdot [x^3]f_{m,1} \\
= & [x^3]f_{3,m} - 2 \cdot \big([x^3]f_{2,m} - 2 \cdot [x^3]f_{1,m}\big) - 2 \cdot \big([x^2]f_{1,m} \cdot [x^1]f_{1,m}\big) - 3 \cdot [x^3]f_{1,m} \\
= & \frac{1}{2}(9m^3 - 63m^2 + 160m - 140) - 2 \cdot \left[2^3 \cdot \binom{m-3+1}{3} - 2 \cdot \binom{m-3+1}{3}\right] \\
& -2 \cdot \binom{m-2+1}{2} \cdot m - 3 \cdot \binom{m-3+1}{3} \\
= & m^3 - 6m^2 + 13m - 10
\end{aligned}$$

Note that this case will not occur when $m \leq 2$.

The second case is shown as below,

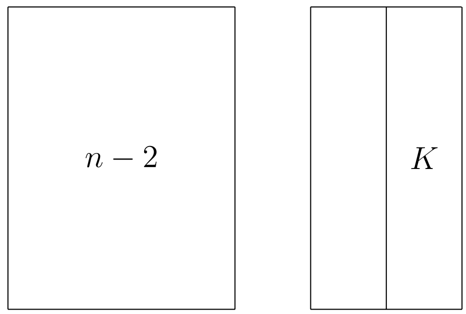


then we must have 1 king on the $m \times (n-3)$ chessboard. Thus $n-3 \geq 1$, so $n \geq 4$ and the number of placements is

$$\begin{aligned}
& [x^1]f_{m,n-3}(x) \cdot \big([x^2]f_{m,2}(x) - 2 \cdot [x^2]f_{m,1}(x)\big) \\
= & [x^1]f_{m,n-3}(x) \cdot \big([x^2]f_{2,m}(x) - 2 \cdot [x^2]f_{1,m}(x)\big) \\
= & m(n-3) \cdot \left[2^2 \cdot \binom{m-1}{2} - 2 \cdot \binom{m-1}{2}\right] \\
= & m(n-3) \cdot 2 \cdot \binom{m-1}{2} \\
= & (m^3 - 3m^2 + 2m)(n-3) \\
= & (m^3 - 3m^2 + 2m)n - 3(m^3 - 3m^2 + 2m)
\end{aligned}$$

Note that this case will not occur when $m \leq 2$.

The third case is shown as below,

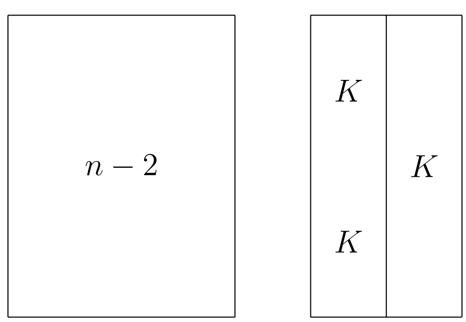


then we must have 2 kings on the $m \times (n-2)$ chessboard. Thus $n-2 \geq 1$, so $n \geq 3$ and the number of placements is

$$\begin{aligned}
& [x^2]f_{m,n-2} \cdot [x^1]f_{m,1} \\
= \; & [x^2]f_{m,n-2} \cdot [x^1]f_{1,m} \\
= \; & \frac{m}{2}\left[m^2(n-2)^2 - 9m(n-2) + 6m + 6(n-2) - 4\right] \\
= \; & \frac{m}{2}\left[m^2n^2 + (-4m^2 - 9m + 6)n + (4m^2 + 24m - 16)\right] \\
= \; & \frac{1}{2}m^3n^2 + \frac{1}{2}(-4m^3 - 9m^2 + 6m)n + \frac{1}{2}(4m^3 + 24m^2 - 16m)
\end{aligned}$$

Note that it is because $n-2 \geq 1$, then we are allowed to apply our theorem on $[x^2]f_{m,n-2}$. Also, if $1 \leq m \leq 2$, then $n-2 \geq 3$, so $n \geq 5$.

The forth case is shown as below,



then we must have 0 kings on the $m \times (n-2)$ chessboard. Thus $n-2 \geq 0$, so $n \geq 2$ and the number of placements is

$$\begin{aligned}
& 1 \cdot \frac{1}{2}\left([x^3]f_{m,2} - 2 \cdot [x^3]f_{m,1}\right) \\
= \;& \frac{1}{2}[x^3]f_{2,m} - [x^3]f_{1,m} \\
= \;& \frac{1}{2} \cdot 2^3 \cdot \binom{m-3+1}{3} - \binom{m-3+1}{3} \\
= \;& 3 \cdot \binom{m-2}{3} \\
= \;& \frac{1}{2}(m^3 - 9m^2 + 26m - 24)
\end{aligned}$$

Note that this case will not occur when $m \leq 4$.

Therefore, for $n \geq 4$

$$\begin{aligned}
[x^3]f_{m,n}(x) \;=\; & [x^3]f_{m,n-1} + \frac{1}{6}(m^3 - 9m^2 + 26m - 24) + \frac{1}{2}(m^3 - 9m^2 + 26m - 24) \\
& + \left[\frac{1}{2}(m^3 - 3m^2 + 2m)n - (m^3 - 3m^2 + 2m)\right] + (m^3 - 6m^2 + 13m - 10) \\
& + \left[(m^3 - 3m^2 + 2m)n - 3(m^3 - 3m^2 + 2m)\right] \\
& + \left[\frac{1}{2}m^3n^2 + \frac{1}{2}(-4m^3 - 9m^2 + 6m)n + \frac{1}{2}(4m^3 + 24m^2 - 16m)\right] \\
& + \frac{1}{2}(m^3 - 9m^2 + 26m - 24) \\
=\; & [x^3]f_{m,n-1} + \frac{1}{6}(m^3 + 45m^2 + 164m - 228) + \frac{1}{2}(-m^3 - 18m^2 + 12m)n + \frac{1}{2}m^3n^2
\end{aligned}$$

Note that if $n = 3$, then we do not have the sixth case. Meanwhile, $(m^3 - 3m^2 + 2m)n - 3(m^3 - 3m^2 + 2m) = 0$. Thus, the above equation holds for $n \geq 3$.

We would like to lower the bound for m. Note that for $m \in \{2, 3, 4\}$ and $n \geq 3$ some of the above cases cannot occur, and the polynomial valued at those m and n is equal to 0.

- If $m = 2$ with $n \geq 6$, then we do not have the second, the third, the forth, the fifth, the sixth, and the last case. Meanwhile, $m^3 - 9m^2 + 26m - 24 = 0$, $m^3 - 3m^2 + 2m = 0$, $m^3 - 6m^2 + 13m - 10 = 0$ and $m^3 - 9m^2 + 26m - 24 = 0$.

- If $m = 2$ with $n = 5$, then we also do not have the first case. Meanwhile, $[x^3]f_{2,4} = 0$.

- If $m = 2$ with $3 \leq n \leq 4$, then we also do not have the seventh case. Meanwhile, $[x^3]f_{2,3} = [x^3]f_{2,4} = 0$ and $m^3n^2 + (-4m^3 - 9m^2 + 6m)n + (4m^3 + 24m^2 - 16m) = 0$.

Thus, the above equation holds for $m \geq 2$ with $n \geq 3$.

Note that when $m = 1$, $m^3 - 9m^2 + 26m - 24 = -6$. The second, the third and the last case will give us negative number of ways, which cannot occur. Thus the lowest bound for m is 2.

Therefore,

$$
\begin{aligned}
&[x^3]f_{m,2} = 2^3 \cdot \binom{m-3+1}{3} \\
&[x^3]f_{m,3} = [x^3]f_{m,2} + \frac{1}{6}(m^3 + 45m^2 + 164m - 228) + \frac{1}{2}(-m^3 - 18m^2 + 12m) \cdot 3 + \frac{1}{2}m^3 \cdot 3^2 \\
&[x^3]f_{m,4} = [x^3]f_{m,3} + \frac{1}{6}(m^3 + 45m^2 + 164m - 228) + \frac{1}{2}(-m^3 - 18m^2 + 12m) \cdot 4 + \frac{1}{2}m^3 \cdot 4^2 \\
&[x^3]f_{m,5} = [x^3]f_{m,4} + \frac{1}{6}(m^3 + 45m^2 + 164m - 228) + \frac{1}{2}(-m^3 - 18m^2 + 12m) \cdot 5 + \frac{1}{2}m^3 \cdot 5^2 \\
&\qquad\vdots \\
&[x^3]f_{m,n} = [x^3]f_{m,n-1} + \frac{1}{6}(m^3 + 45m^2 + 164m - 228) + \frac{1}{2}(-m^3 - 18m^2 + 12m) \cdot n + \frac{m^3}{2} \cdot n^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
[x^3]f_{m,n}(x) &= \frac{4}{3}(m^3 - 9m^2 + 26m - 24) + \frac{1}{6}(m^3 + 45m^2 + 164m - 228)(n-2) \\
&\quad + \frac{1}{2}(-m^3 - 18m^2 + 12m)\left[\frac{n}{2}(n+1) - 3\right] + \frac{m^3}{2}\left[\frac{n}{6}(n+1)(2n+1) - 5\right] \\
&= \frac{1}{6}(m^3n^3 - 27m^2n^2 + 18mn(m+n) + 182mn - 228(m+n) + 264)
\end{aligned}
$$

Note that when $n = 2$, the last three terms vanish. Hence our formula is good for $n = 2$. Therefore,

Theorem 7.6. *For $m \geq 2$ and $n \geq 2$,*

$$[x^3]f_{m,n} = \frac{1}{6}(m^3n^3 - 27m^2n^2 + 18mn(m+n) + 182mn - 228(m+n) + 264).$$

Corollary 7.7. *For $n \geq 2$,*

$$[x^3]f_{2,n} = \frac{1}{6}(8n^3 - 72n^2 + 208n - 192) = 2^3 \cdot \binom{n-3+1}{3}.$$

Note that this formula has been previously obtained in Section 1, proving that this method is correct. This method gives us a recursive and combinatorial proof for the formula of the polynomial coefficient of $f_{m,n}(x)$, but fails to be generalized for $k \geq 5$ (even for $k = 4$ is extremely intricate), since the recursions become too complicated and there are too many cases to consider.

7.2.1 Formula for $[x^4]$ in $f_{5,n}(x)$

Here we compute the coefficient of x^4 in $f_{5,n}(x)$ by the same method as above, showing that this combinatorial numerical method can be generalized but it becomes very inefficient as the number of cases to be analyzed increases.

Note that for a column of length ≤ 4, there are at most 2 kings, while for a column of length ≤ 6, there are at most 3 kings. Thus, we can treat $[x^4]f_{m,n}(x)$ as the special cases for $m \leq 4$ and $4 \leq m \leq 6$ respectively, and discuss $[x^4]f_{m,n}(x)$ for $m \geq 7$.

We construct our formula by considering the number of kings that can be placed in the last column. We list all the cases below.

Case A: If there are 4 kings on the last column.

	$4K$

Case B: If there are 3 kings on the last column.

	$1K$	$3K$

$1K$		$3K$

Case C: If there are 2 kings on the last column.

	$2K$	$2K$

	$1K$	$1K$	$2K$

$1K$		$1K$	$2K$

$2K$		$2K$

Case D: If there is only 1 king on the last column.

	$3K$	$1K$

	$1K$	$2K$	$1K$

$1K$		$2K$	$1K$

	$2K$	$1K$	$1K$

	$1K$	$1K$	$1K$	$1K$

$1K$		$1K$	$1K$	$1K$

$2K$		$1K$	$1K$		$3K$		$1K$

Case E: If there are no kings on the last column.

$4K$	

Let $s \in \{A, B, C, D, E\}$. We will write (s, j) to refer the j-th situation in case s and denote s_j as the number of ways to place 4 kings on the corresponding $m \times n$ chessboard in case (s, j).

Note that for a column of length 5, there are at most 3 kings. Hence, case $(A, 1)$ is impossible. Thus

$$A_1 = 0.$$

Also, if there are 3 kings on some fixed column, then the columns adjacent to that column must contain no kings. Thus case $(B, 1)$ is impossible. Thus

$$B_1 = 0.$$

For case $(B, 2)$, we must have 1 king on the $5 \times (n - 2)$ chessboard. Thus $n - 2 \geq 1$, so $n \geq 3$ and the number of ways is

$$\begin{aligned} B_2 &= [x^1] f_{5,n-2} \times [x^3] f_{5,1} \\ &= 5n - 10. \end{aligned}$$

Furthermore, if there are 2 kings on some fixed column, then the columns adjacent to that column cannot contain 2 kings. Thus case $(C, 1)$ is impossible. Thus

$$C_1 = 0.$$

For case $(C, 2)$, we must have 0 kings on the $5 \times (n - 3)$ chessboard. Thus $n - 3 \geq 0$, so $n \geq 3$ and the number of placements is

$$C_2 = 1 \times 8$$

For case $(C, 3)$, we must have 1 king on the $5 \times (n - 3)$ chessboard. Thus $n - 3 \geq 1$, so $n \geq 4$ and the number of placements is

$$\begin{aligned} C_3 &= [x^1] f_{5,n-3} \cdot 3 \\ &= 15n - 45. \end{aligned}$$

For case $(C, 4)$, we must have 2 kings on the $5 \times (n-2)$ chessboard. Thus $n-2 \geq 1$, so $n \geq 3$ and the number of placements is

$$\begin{aligned} C_4 &= [x^2]f_{5,n-2} \cdot [x^2]f_{5,1} \\ &= \frac{1}{2}\left[25(n-2)^2 - 39(n-2) + 26\right] \cdot 6 \\ &= 75n^2 - 417n + 612. \end{aligned}$$

Note that since $n-2 \geq 1$, then we can apply the formula.

Similarly, case $(D, 1)$ is impossible, either. Thus

$$D_1 = 0.$$

For case $(D, 2)$, we must have 0 kings on the $5 \times (n-3)$ chessboard. Thus $n-3 \geq 0$, so $n \geq 3$ and the number of placements is

$$D_2 = 1 \times 3.$$

For case $(D, 3)$, we must have 1 king on the $5 \times (n-3)$ chessboard. Thus $n-3 \geq 1$, so $n \geq 4$ and the number of placements is

$$\begin{aligned} D_3 &= [x^1]f_{5,n-3} \times 3 \\ &= 15n - 45. \end{aligned}$$

For case $(D, 4)$, we must have 0 kings on the $5 \times (n-3)$ chessboard. Thus $n-3 \geq 0$, so $n \geq 3$ and the number of placements is

$$D_4 = 1 \times 8.$$

For case $(D, 5)$, we must have 0 kings on the $5 \times (n-4)$ chessboard. Thus $n-4 \geq 0$, so $n \geq 4$ and the number of placements is

$$D_5 = 1 \times 74.$$

For case $(D, 6)$, we must have 1 king on the $5 \times (n-4)$ chessboard. Thus $n-4 \geq 1$, so $n \geq 5$ and the number of placements is

$$\begin{aligned} D_6 &= [x^1]f_{5,n-4} \times 30 \\ &= 150n - 600. \end{aligned}$$

For case $(D, 7)$, we must have 2 king on the $5 \times (n-3)$ chessboard. Thus $n-3 \geq 1$, so $n \geq 4$ and the number of placements is

$$\begin{aligned}
D_7 &= [x^2] f_{5,n-3} \cdot 12 \\
&= \frac{1}{2}\left[25(n-3)^2 - 39(n-3) + 26\right] \cdot 12 \\
&= 150n^2 - 1134n + 2208.
\end{aligned}$$

Note that since $n-3 \geq 1$, then we can apply the formula.

For case $(D, 8)$, we must have 3 king on the $5 \times (n-2)$ chessboard. Thus $n-2 \geq 1$, so $n \geq 3$ and the number of placements is

$$\begin{aligned}
D_8 &= [x^3] f_{5,n-2} \cdot [x^1] f_{5,1} \\
&= \frac{1}{6}\left[125(n-2)^3 - 585(n-2)^2 + 1132(n-2) - 876\right] \cdot 5 \\
&= \frac{5}{6}\left[125n^3 - 1335n^2 + 4972n - 6480\right].
\end{aligned}$$

Note that here we force $n-2 \geq 2$ in order to use the formula, so $n \geq 4$.

Finally,for case $(E, 1)$, we must have 4 king on the $5 \times (n-1)$ chessboard. Thus $n-1 \geq 3$, so $n \geq 4$ and the number of placements is

$$E_1 = [x^4] f_{5,n-1} \times 1.$$

Therefore, for $n \geq 5$,

$$\begin{aligned}
[x^4] f_{5,n} \quad &= \quad (5n-10) + 8 + (15n-45) + (75n^2 - 417n + 612) + 3 + (15n-45) \\
&\quad +8 + 74 + (150n - 600) + (150n^2 - 1134n + 2208) \\
&\quad +\frac{5}{6}\left[125n^3 - 1335n^2 + 4972n - 6480\right] + [x^4] f_{5,n-1} \\
&= \quad [x^4] f_{5,n-1} + \frac{625}{6}n^3 - \frac{1775}{2}n^2 + \frac{8332}{3}n - 3187.
\end{aligned}$$

Note that if $n = 4$, then the case $(D, 6)$ does not occur. Meanwhile, $D_6 = 150n - 600 = 0$. Thus, the above equation holds for $n \geq 4$.

Now

$$\begin{aligned}
[x^4]f_{5,3} &= 65\\
[x^4]f_{5,4} &= [x^4]f_{5,3} + \frac{625}{6}\cdot 4^3 - \frac{1775}{2}\cdot 4^2 + \frac{8332}{3}\cdot 4 - 3187\\
[x^4]f_{5,5} &= [x^4]f_{5,4} + \frac{625}{6}\cdot 5^3 - \frac{1775}{2}\cdot 5^2 + \frac{8332}{3}\cdot 5 - 3187\\
[x^4]f_{5,6} &= [x^4]f_{5,5} + \frac{625}{6}\cdot 6^3 - \frac{1775}{2}\cdot 6^2 + \frac{8332}{3}\cdot 6 - 3187\\
&\vdots\\
[x^4]f_{5,n} &= [x^4]f_{5,n-1} + \frac{625}{6}\cdot n^3 - \frac{1775}{2}\cdot n^2 + \frac{8332}{3}\cdot n - 3187
\end{aligned}$$

Thus

$$\begin{aligned}
[x^4]f_{5,n} &= 65 + \frac{625}{6}\cdot\left[\frac{n^2}{4}(n+1)^2 - 36\right] - \frac{1775}{2}\left[\frac{n}{6}(n+1)(2n+1) - 14\right]\\
&\quad + \frac{8332}{3}\cdot\left[\frac{n}{2}(n+1) - 6\right] - 3187\cdot(n-3)\\
&= \frac{1}{24}(625n^4 - 5850n^3 + 23303n^2 - 46710n + 39288).
\end{aligned}$$

Note that when $n = 3$, the last four terms vanish. Hence our formula is good for $n = 3$. Therefore,

Theorem 7.8. *For* $n \geq 3$*,*

$$[x^4]f_{5,n} = \frac{1}{24}(625n^4 - 5850n^3 + 23303n^2 - 46710n + 39288).$$

7.3 Second Alternative Method - Recurrence Relations

This method uses the recursive relations for the generating function of $f_{m,n}$, which is very efficient but has the disadvantage that we have to know the recurrence relation beforehand. Since we only computed them up to $f_{6,n}$, this method fail for larger values of m.

It is known that, for $n \geq 4$,

$$f_{3,n}(x) = (1+x)f_{3,n-1}(x) + (2x + x^2)f_{3,n-2}(x) - (x^2 + x^3)f_{3,n-3}(x).$$

Therefore,

$$\begin{aligned}
[x^2]f_{3,n} &= [x^2](1+x)f_{3,n-1} + [x^2](2x+x^2)f_{3,n-2} - [x^2](x^2+x^3)f_{3,n-3}\\
&= [x^2]f_{3,n-1} + [x^1]f_{3,n-1} + 2\times[x^1]f_{3,n-2} + [x^0]f_{3,n-2} - [x^0]f_{3,n-3} - [x^{-1}]f_{3,n-3}\\
&= [x^2]f_{3,n-1} + 3(n-1) + 2\times 3(n-2) + 1 - 1 - 0\\
&= [x^2]f_{3,n-1} + 9n - 15
\end{aligned}$$

We obtain the same recurrence relation of $[x^2]f_{3,n}$ as we did in the first approach. Moreover, since

$$
\begin{aligned}
f_{3,1}(x) &= 1 + 3x + x^2 \\
f_{3,2}(x) &= 1 + 6x + 4x^2 \\
f_{3,3}(x) &= 1 + 9x + 16x^2 + 8x^3 + x^4
\end{aligned}
$$

we discover that

$$
\begin{aligned}
[x^2]f_{3,2} &= [x^1]f_{3,1} + 3 \\
&= [x^1]f_{3,1} + 9 \times 2 - 15 \\
[x^2]f_{3,3} &= [x^2]f_{3,2} + 12 \\
&= [x^1]f_{3,2} + 9 \times 3 - 15
\end{aligned}
$$

Thus our recurrence relation holds for $n \geq 2$. The rest proof is the same as in the previous section.

7.4 Generalization for the Leading Term

Previously, we conjectured that the coefficient of x^k in $f_{m,n}(x)$, denoted by $P_k(m,n)$, is a symmetric polynomial in m and n of total degree $2k$.

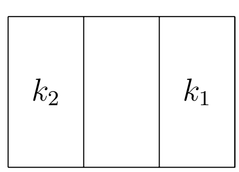
Theorem 7.9. *The leading term in $P_k(m,n)$ is*

$$\frac{m^k n^k}{k!}$$

Proof. We will use induction on k to prove the above conjecture. Therefore, suppose the above assertions hold for $\leq k-1$ (the base case is trivial).

To compute $[x^k]f_{m,n}$, we consider the distribution of kings in the last column compared to the other columns.

Take the case when there are k_1 kings on the last column and no kings on the column adjacent to it. Then in the first $n-2$ columns, we must have $k_2 = k - k_1$ kings.



The number of placements is

$$[x^{k_2}]f_{m,n-2} \times [x^{k_1}]f_{m,1},$$

which is a polynomial with degree $k = k_2 + k_1$ in m, but only k_2 in n.

Note that in the following case,

k_2	k_1

the number of placements is

$$[x^{k_2+k_1}]f_{m,2} - \epsilon(x)$$

where $\epsilon(x)$ denotes the error terms. Note that the error terms is also a polynomial with degree no more than $k_2 + k_1$ in m.

Therefore, in these cases similar to the following

k_3		k_2	k_1

the number of placements is no more than

$$[x^{k_3}]f_{m,n-2} \cdot [x^{k_2+k_1}]f_{m,2}$$

which is a polynomial with degree $k = k_3 + k_2 + k_1$ in m, but only k_3 in n.

Therefore, we want to maximize $k_{max(i)}$. Suppose $k_{max(i)} = k$, then we have no kings on the last column. Then we must have k kings on the first $n-1$ columns.

k	

The number of placements is

$$[x^k]f_{m,n-1}$$

However, we will cancel this term later when summing all the recurrence equations up.

Therefore, our next choice is $k_{max(i)} = k-1$ Suppose we have only 1 king on the last column and have no kings on the column next to the last column. Then we must have $k-1$ kings on the first $n-2$ columns.

$k-1$		1

The number of placements is

$$[x^{k-1}]f_{m,n-2} \cdot [x]f_{m,1}$$

which is a polynomial with degree k in m and $k-1$ in n.

When summing all the recurrence equations up, since

$$\sum n^{k-1} = \frac{n^k}{k} + \cdots .$$

Therefore, by induction hypothesis, the leading coefficient of

$$\sum \left([x^{k-1}]f_{m,n-2} \cdot [x]f_{m,1}\right)$$

is the leading coefficient of

$$m \sum \frac{m^{k-1}n^{k-1}}{(k-1)!} = \frac{m^k}{(k-1)!} \sum n^{k-1}$$

is $\frac{m^k n^k}{k!}$ □

7.5 Other Methods and Ideas

So far, to proceed with the double interpolation, we compute $(k+1)^2$ powers of the adjacent matrices: $A^{2k-1}_{2k-1}, A^{2k}_{2k-1}, \cdots, A^{3k-1}_{2k-1}, A^{2k-1}_{2k}, \cdots, A^{3k-1}_{2k}, \cdots, A^{2k-1}_{3k-1}, A^{2k}_{3k-1}, \cdots, A^{3k-1}_{3k-1}$. We can improve this by computing instead only half of these values and completing the $(k+1)\cdot(k+1)$ matrix by symmetry, being easier to compute A^n_m than A^m_n for $m < n$ and they are equal.

7.6 Congruences

Because so far we have not been able to find the closed formula for the coefficient of x^k in $f_{m,n}(x)$, we started to look for congruences modulo the size of the boards. We intend to use these to find some patterns and a method of computing the closed formula for the polynomial using

We have the result that for $m \geq 1$ and $n \geq 1$,

$$[x^2](f_{m,n}(x)) = \frac{1}{2}(m^2n^2 - 9mn + 6(m+n) - 4).$$

Theorem 7.10. *For $t \geq 0$ and $n \geq 1$, we have the following pattern:*

$$\begin{aligned}
[x^2]f_{2+12t,n} &= 0 \mod (2) \\
[x^2]f_{3+12t,n} &= 1 \mod (3) \\
[x^2]f_{6+12t,n} &= 4 \mod (6) \\
[x^2]f_{9+12t,n} &= 1 \mod (3) \\
[x^2]f_{10+12t,n} &= 0 \mod (2) \\
[x^2]f_{12t,n} &= 1 \mod (3)
\end{aligned}$$

Proof.

- $m = 12t + 2$

$$\begin{aligned}
[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= 72n^2t^2 + 24n^2t + 2n^2 - 54nt - 6n + 36t + 4 \equiv 0 \mod (2)
\end{aligned}$$

- $m = 12t + 3$

$$\begin{aligned}
[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= 72n^2t^2 + 36n^2t + \frac{9}{2}n^2 - 54nt - \frac{21}{2}n + 36t + 7 \equiv 1 \mod (3)
\end{aligned}$$

since $n(9n - 21)$ is even.

- $m = 12t + 6$

$$\begin{aligned}
[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= 72n^2t^2 + 72n^2t + 18n^2 - 54nt - 24n + 36t + 16 \equiv 4 \mod (6)
\end{aligned}$$

- $m = 12t + 9$

$$\begin{aligned}[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right)\\ &= 72n^2t^2 + 108n^2t + \frac{81}{2}n^2 - 54nt - \frac{75}{2}n + 36t + 25 \equiv 1 \mod (3)\end{aligned}$$

since $n(81n - 75)$ is even.

- $m = 12t + 10$

$$\begin{aligned}[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right)\\ &= 72n^2t^2 + 120n^2t + 50n^2 - 54nt - 42n + 36t + 28 \equiv 0 \mod (2)\end{aligned}$$

- $m = 12t$

$$\begin{aligned}[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right)\\ &= 72n^2t^2 - 54nt + 3n + 36t - 2 \equiv 1 \mod (3)\end{aligned}$$

□

Theorem 7.11. *For $t \geq 1$ and $n \geq 0$,*

$$[x^2]f_{3t,n} \equiv 1 \mod 3$$

Proof.

$$\begin{aligned}[x^2](f_{3t,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right)\\ &= \frac{1}{2}\left(9t^2n^2 - 27tn + 6(3t+n) - 4\right)\\ &= \frac{1}{2}\left(9tn(tn-3)\right) + 9t + 3n - 2\\ &\equiv 1 \mod (3)\end{aligned}$$

□

Theorem 7.12. *For* $n \geq 2k^2 + k + 2$*,*

$$[x^2]f_{2k+2,n} \equiv 2k^2 + k \mod (n-1)$$

Proof. Let $m = 2k + 2$

$$\begin{aligned}
[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= \frac{1}{2}\left((2k+2)^2n^2 - 9(2k+2)m + 6(2k+2+n) - 4\right) \\
&= (2k^2+4k+2)n^2 - (9k+9)n + 6k + 6 + 3n - 2 \\
&= (2k^2+4k+2)(n-1)^2 + (n-1)(4k^2-k-2) + 2k^2 + k \\
&\equiv 2k^2 + k \mod (n-1)
\end{aligned}$$

as we need $2k^2 + k < n - 1 \Rightarrow n \geq 2k^2 + k + 2$. □

Theorem 7.13. *For* $t \geq 2k^2 - k + 1$*,*

$$[x^2]f_{2k+1,2t+1} \equiv t + 2k^2 - k \mod (2t)$$

Proof. Let $m = 2k + 1$ and $n = 2t + 1$.

$$\begin{aligned}
[x^2](f_{m,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= \frac{1}{2}\left((2k+1)^2(2t+1)^2 - 9(2k+1)(2t+1) + 6(2k+2t+2) - 4\right) \\
&= 8k^2t^2 + 8k^2t + 8kt^2 + 2t^2 - 10kt - t + 2k^2 - k \\
&\equiv t + 2k^2 - k \mod (2t), t \geq 2k^2 - k + 1
\end{aligned}$$

since we need $2t \geq t + 2k^2 - k + 1 \Rightarrow t \geq 2k^2 - k + 1$. □

Theorem 7.14. *For* $n \geq 8k^2 + 4k + 3$

$$[x^2](f_{2k+2,n}(x)) = 8k^2 + 4k \mod (n-2)$$

Proof.

$$\begin{aligned}
[x^2](f_{2k+2,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= (2k^2+4k+2)(n^2-4n+4) + 8nk^2 + 8n + 16nk - 8k^2 - 8 - 16k - 9nk - 6n + 6k + 4 \\
&= (2k^2+4k+2)(n-2)^2 + n(8k^2+7k+2) - 8k^2 - 10k - 4 \\
&= (2k^2+4k+2)(n-2)^2 + (n-2)(8k^2+7k+2) + 16k^2 + 14k + 4 - 8k^2 - 10k - 4 \\
&= (2k^2+4k+2)(n-2)^2 + (n-2)(8k^2+7k+2) + 8k^2 + 4k \\
&\equiv 8k^2 + 4k \mod (n-2), \forall n \geq 8k^2 + 4k + 3, t \geq 8k^2 - 4k + 2
\end{aligned}$$

since we need $n - 2 \geq 8k^2 + 4k + 1 \Rightarrow n \geq 8k^2 + 4k + 3$. □

Theorem 7.15. *For* $t \geq 8k^2 - 4k + 2$

$$[x^2](f_{2k+1,2t}(x)) = t + 8k^2 - 4k - 1 \mod (2t-2)$$

Proof.

$$\begin{aligned}
[x^2](f_{2k+1,n}(x)) &= \frac{1}{2}\left(m^2n^2 - 9mn + 6(m+n) - 4\right) \\
&= (2k+1)^2 2t^2 - 9t(2k+1) + 3(2k+1+2t) - 2 \\
&= (4k^2+4k+1)(2t^2-2t) + 8tk^2 + 8tk + 2t - 18tk - 9t + 6k + 3 + 6t - 2 \\
&= (4k^2+4k+1)t(2t-2) + 2t(4k^2-5k-1) + t + 6k + 1 \\
&= (4k^2+4k+1)t(2t-2) + (2t-2)(4k^2-5k-1) + 8k^2 - 10k - 2 + t + 6k + 1 \\
&= (4k^2+4k+1)t(2t-2) + (2t-2)(4k^2-5k-1) + t + 8k^2 - 4k - 1 \\
&\equiv t + 8k^2 - 4k - 1 \mod (2t-2), \forall t \geq 8k^2 - 4k + 2
\end{aligned}$$

since we need $2t - 2 \geq t + 8k^2 \Rightarrow t \geq$.

Observe that in this formula also depends on t, not only on k.

□

Theorem 7.16. *For* $t \geq 4k^2 - 2k + 1$

$$[y^2](f_{2k+1,2t+1}(y)) = -4k \mod (2t-1)$$

Proof.

$$\begin{aligned}
[x^2](f_{2k+1,2t+1}(x)) &= \frac{1}{2}(m^2n^2 - 9mn + 6(m+n) - 4) \\
&= \frac{1}{2}\left((2k+1)^2(2t+1)^2 - 9(2k+1)(2t+1) + 6(2k+2t+2) - 4\right) \\
&= 8k^2t^2 + 8k^2t + 8kt^2 + 2k^2 - 10kt + 2t^2 - k - t \\
&= (2t^2-t)(4k^2+4k+1) + 12tk^2 + 2k^2 - 6tk - k \\
&= (2t^2-t)(4k^2+4k+1) + (2t-1)(6k^2-3k) - 4k \\
&\equiv 8k^2 - 4k \mod (2t-1), t \geq 4k^2 - 2k + 1,
\end{aligned}$$

since we need $2t - 1 \geq 8k^2 - 4k \Rightarrow t \geq 4k^2 - 2k + 1$. □

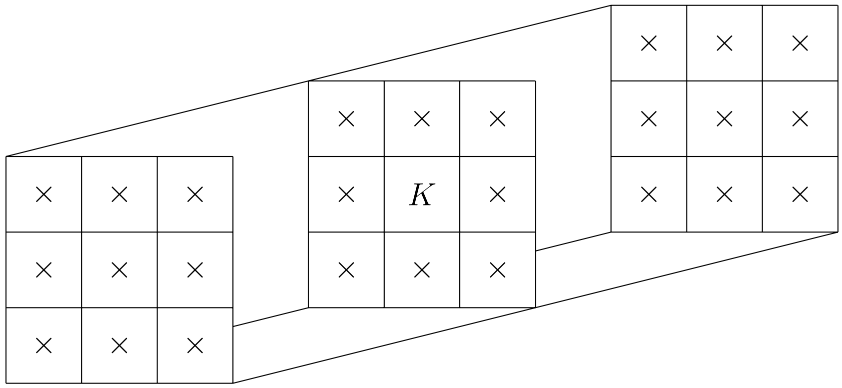
We would like to generalize for $k = 3$ and beyond, and try to use this combinatorial method to generate the closed formula for the coefficients of $f_{m,n}(x)$.

8 3D Generalization

8.1 Introduction

Let $f_{p,q,r}$ denote the number of placements of non attacking kings on a 3 dimensional $p \times q \times r$ board.

In a 3 dimensional chessboard, a king can attack the following places.



Define the generating function for the $p \times q \times r$ boards to be

$$f_{p,q,r}(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where a_k denotes the number of ways to place k kings on an $p \times q \times r$ chessboard. Note that

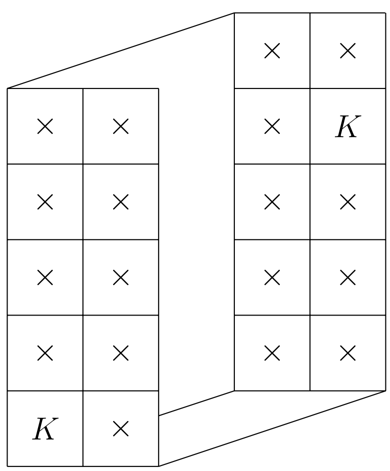
$$f_{p,q,r}(x) = f_{p,r,q}(x) = f_{q,p,r}(x) = f_{q,r,p}(x) = f_{r,p,q}(x) = f_{r,q,p}(x),$$

and

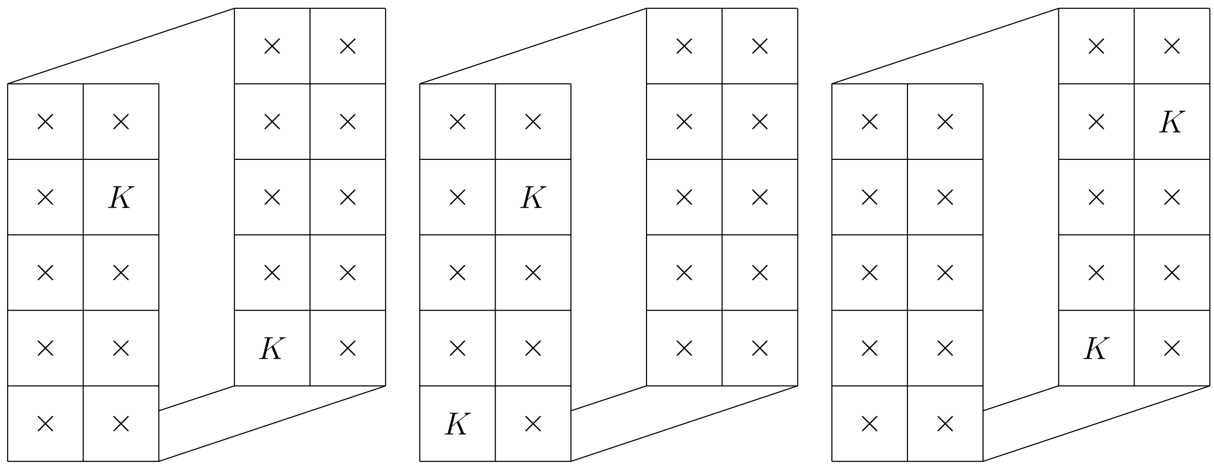
$$f_{p,q,1}(x) = f_{p,q}(x),$$

which is the number of configuration of kings on a two dimensional $p \times q$ chessboard.

Now, we start by observing the following example.



In fact, the above case is "equivalent" to the following cases.



Therefore, for each king on

×	×
×	K
×	×
×	×
K	×

there are 2 places (front board or back board) to choose.
Therefore, we have

Conjecture 8.1.

$$f_{p,q,2}(x) = f_{p,q,1}(2x).$$

Conjecture 8.2.

$$f_{d_1,d_2,\cdots,d_n,2}(x) = f_{d_1,d_2,d_3,\cdots,d_n}(2x).$$

8.2 Bounds

We have the following theorem:

Theorem 8.3.

$$\max_{i,j\in\{1,2,3\}} \left[\left(f_{d_i d_j}^{\frac{1}{d_i d_j}} \right)^{\frac{1}{2}} \right] < f_{d_1,d_2,d_3}^{\frac{1}{d_1 d_2 d_3}} \le \min_{i,j\in\{1,2,3\}} \left(f_{d_i d_j}^{\frac{1}{d_i d_j}} \right)$$

Proof. For an $p \times q \times r$ chessboard, we can regard it as an $p \times q$ chessboard with r layers. For each $p \times q$ chessboard, there are $f_{p,q}$ ways to place kings on the chessboard, if we ignore the non attacking kings condition. Thus, there are total $(f_{p,q})^r$ ways to place kings on these r chessboards, where they may or may not attack each other. Of course that imposing the non attacking condition will make this number an upper bound for our number of configurations. Hence

$$f_{p,q,r} \leq (f_{p,q})^r$$

Note that the equality holds only when $r = 1$. Therefore,

$$f_{p,q,r}^{\frac{1}{pqr}} \leq \left(f_{p,q}^r\right)^{\frac{1}{pqr}} = f_{p,q}^{\frac{1}{pq}}$$

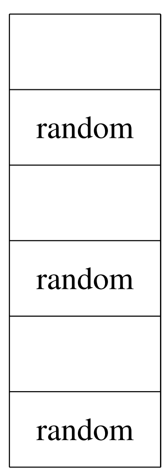
Since $f_{p,q,r}$ is symmetric, so

$$f_{d_1,d_2,d_3}^{\frac{1}{d_1 d_2 d_3}} \leq \min_{i,j \in \{1,2,3\}} \left(f_{d_i d_j}^{\frac{1}{d_i d_j}} \right)$$

Now we want to find the lower bound.

Case 1 For r even $\Rightarrow r = 2k$.

For an $p \times q \times 2k$ chessboard, we can regard it as an $p \times q \times 2$ chessboard with k layers. For each $p \times q \times 2$ chessboard, we can compute the number of configurations for a random $p \times q$ chessboard at the bottom and an empty $p \times q$ chessboard at the top, shown as below, which will give us a lower bound on our number, since this is a particular case.



There are $f_{p,q}$ ways to place kings on the bottom chessboard and 1 way to place kings on the top chessboard. Thus, there are total $(f_{p,q})^k$ ways to place kings on these $2k$ chessboards. Hence

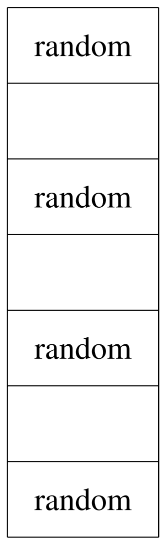
$$f_{p,q,2k} > (f_{p,q})^k$$

Therefore,

$$f_{p,q,r}^{\frac{1}{pqr}} = f_{p,q,2k}^{\frac{1}{pq(2k)}} > \left(f_{p,q}^{k}\right)^{\frac{1}{pq(2k)}} = \left(f_{p,q}^{\frac{1}{pq}}\right)^{\frac{1}{2}}$$

Case 2 For r odd $\Rightarrow r = 2k+1$.

For an $p \times q \times (2k+1)$ chessboard, we can regard it as the above case plus an additional random $p \times q$ chessboard at the very top, shown as below. Counting the number of configurations for the model in the picture, similar to the first case, will give us again a lower number on our number, since we are considering a particular case with an extra condition.



Thus, there are total $(f_{p,q})^{k+1}$ ways to place kings on these $2k+1$ chessboards. Hence

$$f_{p,q,2k+1} > (f_{p,q})^{k+1}$$

Therefore,

$$f_{p,q,r}^{\frac{1}{pqr}} = f_{p,q,2k+1}^{\frac{1}{pq(2k+1)}} > \left(f_{p,q}^{k+1}\right)^{\frac{1}{pq(2k+1)}} = \left(f_{p,q}^{\frac{1}{pq}}\right)^{\frac{k+1}{2k+1}} = \left(f_{p,q}^{\frac{1}{pq}}\right)^{\frac{k+\frac{1}{2}}{2k+1}+\frac{\frac{1}{2}}{2k+1}} > \left(f_{p,q}^{\frac{1}{pq}}\right)^{\frac{1}{2}}$$

Since $f_{p,q,r}$ is symmetric, we have

$$f_{d_1,d_2,d_3}^{\frac{1}{d_1 d_2 d_3}} > \max_{i,j\in\{1,2,3\}} \left[\left(f_{d_i d_j}^{\frac{1}{d_i d_j}}\right)^{\frac{1}{2}}\right]$$

$\square$

The following example matches our theorem.

Example 8.4. *We know*

$$f_{2,3,4} = 589 \Rightarrow f_{2,3,4}^{\frac{1}{24}} = 1.30443$$

Since

$$\begin{aligned}
&f_{2,3} = 11, f_{2,4} = 21, f_{3,4} = 93 \Rightarrow \\
&f_{2,3}^{\frac{1}{6}} = 1.49130, f_{2,4}^{\frac{1}{8}} = 1.46311, f_{3,4}^{\frac{1}{12}} = 1.45895 \\
&f_{2,3}^{\frac{1}{12}} = 1.22119, f_{2,4}^{\frac{1}{16}} = 1.20959, f_{3,4}^{\frac{1}{24}} = 1.20787
\end{aligned}$$

Therefore, we can see that

$$\max(1.22119, 1.20959, 1.20787) < f_{2,3,4}^{\frac{1}{24}} = 1.30443 < \min(1.49130, 1.46311, 1.45895)$$

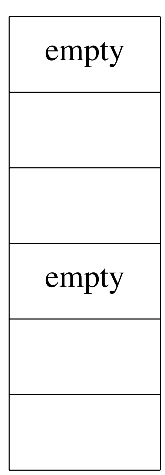
We intend to improve our bounds. One idea is to consider the $p \times q \times 2k$ chessboard an $p \times q \times 2$ chessboard with k layers. For each $p \times q \times 2$ chessboard, there are $f_{p,q,2}$ ways to place kings on the chessboard. Thus, there are total $(f_{p,q,2})^k$ ways to place kings on these $2k$ chessboards. Hence

$$f_{p,q,2k} < (f_{p,q,2})^k .$$

We would like to study if this is a better bound, and if it can be obtained for the case $r = 2k + 1$.

Another idea is to divide the problem into three cases, $r \equiv 0, 1, 2 \mod (3)$.

First, assume that $r = 3k$. For an $p \times q \times 3k$ chessboard, we can regard it as an $p \times q \times 3$ chessboard with k layers. For each $p \times q \times 3$ chessboard, we regard it as a random $p \times q \times 2$ chessboard at the bottom and an empty $p \times q$ chessboard at the top, shown as below.



There are $f_{p,q,2}$ ways to place kings on the bottom chessboard and 1 way to place kings on the top chessboard. Thus, there are total $(f_{p,q,2})^k$ ways to place kings on these $3k$ chessboards. Hence

$$f_{p,q,3k} > (f_{p,q,2})^k$$

Therefore,

$$\begin{aligned} f_{p,q,r}^{\frac{1}{pqr}} &= f_{p,q,3k}^{\frac{1}{pq(3k)}} \\ &> \left(f_{p,q,2}^{k}\right)^{\frac{1}{pq(3k)}} \\ &= \left(f_{p,q,2}^{\frac{1}{pq}}\right)^{\frac{1}{3}} \end{aligned}$$

Applying the same method for $r = 3k + 1$ and $r = 3k + 2$, we intend to analyze if this method gives us better bounds for the number of configurations on 3-dimensional boards.

Lastly, a good bound for $f_{m,n}$ in 2 dimensions is given in [2]:

Theorem 8.5.

$$(f_{m,n})^k < f_{m,(n+1)k} < (f_{m,n+1})^k$$

Using standard methods, we can deduce better bounds for the $3D$ case from here and we can prove the existence of the entropy in three dimensions, obtaining:

Theorem 8.6.

$$(f_{p,q,r})^k < f_{p,q,(r+1)k} < (f_{p,q,r+1})^k$$

8.3 Generate 3D Boards - Idea

We intend to write a code in $Sage$ that will generate the $3D$ boards in the following way:

Consider a list, $List_1$, with all the possible $n \times 1$ columns with kings on them, for a fixed n. The columns are $n \times 1$ matrices in which 1 represents a king and 0 represents an empty cell. Now, by traversing the list, create the $n \times 2$ matrices by deciding if every pair in $List_1$ has attacking kings on it, and if it does not have, adjoin them and append the new matrix to a new list, $List_2$. Repeat the procedure by making adjacent every element in $List_2$ with every column in $List_1$ (adjacent to the right, since it is not necessary to consider them adjacent both to the right and to the left) until we reach $List_m$, which will contain all the $n \times m$ boards with non attacking kings on them in a specific order, for a fixed m.

Now, traverse $List_m$, comparing each pair an $m \times n$ board to see if there will be pairs of attacking kings. If there are not, adjoin them on the third dimension (overlap them) and append them to a new list $List_{3,2}$. Repeat the procedure in the similar way we did for the 2-dimensional case, until we reach a list that contains $n \times m \times k$ matrices. Instead of this, we can also create the adjacency matrix for the $3D$ case: go back to traversing $List_m$, and for each pair, if it is non attacking, create an empty adjacency matrix of dimensions $f_{m,n} \times f_{m,n}$ and put a 1 in the specific cell of the two boards that we were comparing. This adjacency matrix can be used to compute the number of possible $3D$ non attacking kings configuration.

9 Future Work

- Generalize the recursive formula for $f_{m,n}(x)$.
- Write efficient code for calculating powers of adjacency matrices.
- Find patterns and closed formulas for $[x^k](f_{m,n}^{\frac{1}{mn}}(x))$ for general m,n.
- Generalize the Zeckendorf board generation to $3D$ chessboards and higher dimensions.
- What is the relation between the kings' density and the entropy?
- We know that
$$f_{m,n}^{\frac{1}{mn}} \to \eta$$
as $m, n \to \infty$.
η is known to some number of digits. So $f_{m,n}$ grows like η^{mn}. We think that a better approximation of $f_{m,n}$ would be
$$c \cdot \eta^{mn} \cdot \alpha^{m+n},$$

in which we would like to estimate the following constants: η, α, β, c.
We also intend to study the particular case $m = n$, for which the approximation is

$$\frac{f_{n,n}}{\eta^{n^2}} \simeq \alpha^{2(n+n)} n^{2\beta} c.$$

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