2-tone Colorings in Graph Products

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Abstract

A variation of graph coloring known as a t-tone k-coloring assigns a set of t colors to each vertex of a graph from the set $\{1, \cdots, k\}$ where the sets of colors assigned to any two vertices distance d apart share fewer than d colors in common. The minimum integer k such that a graph G has a t-tone k-coloring is known as the t-tone chromatic number. We study the 2-tone chromatic number in three different graph products. In particular, given graphs G and H, we bound the 2-tone chromatic number for the direct product $G \times H$, the Cartesian product $G \square H$, and the strong product $G \boxtimes H$.

1 Introduction

Many variations of classic graph k-colorings abound, whereby we take a k-coloring of a graph to mean an assignment of an element from $\{1, \cdots, k\}$, called a color, to each of the vertices of the graph. Gary Chartrand was the first to introduce a t-tone k-coloring, which is an assignment of t elements from the set $\{1, \cdots, k\}$ to each vertex such that the sets of colors assigned to any two distinct vertices within distance d share fewer than d colors. This t-tone k-coloring variation can be viewed as a generalization of classic graph coloring since a 1-tone k-coloring of a graph is simply a k-coloring.

In an initial investigation of this graph parameter, Fonger, Goss, Phillips, and Segroves [4] determined the exact value for the 2-tone chromatic number of a tree and a complete multipartite graph. In 2011, Bickle and Phillips [2] gave general bounds for the t-tone chromatic number of a graph G in terms of the maximum degree of G. Cranston, Kim, and Kinnersley [3] improved upon these bounds shortly thereafter. In [1], the problem of determining bounds of the t-tone chromatic number of the Cartesian product of two graphs is posed. In this paper, we determine bounds on the 2-tone chromatic number in the direct product, Cartesian product, and the strong product of two graphs.

1.1 Definitions and Notation

An *undirected graph* $G = (V, E)$ is a set of vertices $V(G)$ together with a set of edges $E(G)$ which are unordered pairs (x, y), or simply xy, of vertices of G. If $xy \in E(G)$, we say that x and y are *adjacent*. For the purpose of this paper, we consider only *simple* undirected graphs, meaning undirected graphs that have no loops.

For any vertex $x \in V(G)$, the *open neighborhood of* x, denoted $N(x)$, is defined to be the set of all vertices adjacent to x. The *closed neighborhood of* x is defined to be $N[x] = N(x) \cup \{x\}$. The *degree of* x, denoted $\deg_G(x)$, is the cardinality of the open neighborhood of x. We let $\Delta(G) = \max_{x \in V(G)} \deg_G(x)$. A *path of length* k, denoted P_k, in G between two vertices v_0 and v_k is a sequence $v_0, \cdots, v_k$ of $k+1$ distinct vertices of G such that for all $i \in \{0, \cdots, k-1\}$, the edge $v_i v_{i+1} \in E(G)$. The *distance* between vertices u and v, denoted $d_G(u, v)$ is the smallest length path between u and v. When the context is clear, we use the shorthand notation $d(u, v)$.

As stated before, a *proper* k*-coloring* of a graph G is an assignment of an element from $\{1, \cdots, k\}$, called a *color*, to each vertex in $V(G)$ such that no two adjacent vertices are assigned the same color. The *chromatic number of* G, denoted $\chi(G)$, is the minimum number k such that G has a proper k-coloring. We use K_n to denote the *complete graph on* n *vertices*, which is a graph with n vertices where each vertex has degree $n-1$. Given a graph G, a *clique* is any complete subgraph of G, and the *clique number of* G, denoted $\omega(G)$, is the cardinality of the maximum clique of G. For positive integers t and k where $t \leq k$, we let $[k]$ represent the set $\{1, \cdots, k\}$ and denote the family of t-element subsets of $[k]$ by $\binom{[k]}{t}$. We now give a formal definition of a t-tone k-coloring of a graph.

Definition 1. *Let G be a graph and let t and k be positive integers such that $t \leq k$. A* ***t-tone k-coloring*** *of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(u) \cap f(v)| < d_G(u, v)$ for all distinct vertices u and v. A graph that has a t-tone k-coloring is said to be* ***t-tone k-colorable****. The* **t-tone chromatic number** *of G, denoted $\tau_t(G)$, is the minimum k such that G is t-tone k-colorable.*

Given a t-tone k-coloring f of G, we call $f(v)$ the *label* of v and the elements of $[k]$ *colors*. Figure 1 depicts a 2-tone 5-coloring of P_5 which is, indeed, minimum.

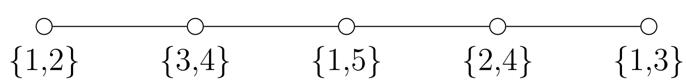

Figure 1: A 2-tone 5-coloring of P_5

In this paper, we focus on 2-tone k-colorings of three different graph products. First, we consider the direct product.

Definition 2. *Given two graphs G and H, the* ***direct product*** *of G and H, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is*

$$E(G \times H) = \{(x_1, y_1)(x_2, y_2) \mid x_1 x_2 \in E(G) \text{ and } y_1 y_2 \in E(H)\}.$$

Figure 2 depicts the direct product of K_2 and K_3.

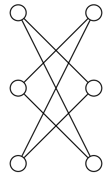

Figure 2: $K_2 \times K_3$

The next graph we consider is the Cartesian product, which is the motivation behind this research. We aim to improve upon the upper bound for the 2-tone chromatic number of the Cartesian product of two graphs given in [1].

Definition 3. *The* ***Cartesian product*** *of graphs G and H, denoted $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$ with edge set*

$$E(G \square H) = \{(x_1, y_1)(x_2, y_2)|\ x_1 y_1 \in E(G) \text{ and } x_2 = y_2, \text{ or } x_1 = y_1 \text{ and } x_2 y_2 \in E(H)\}.$$

Figure 3 depicts the Cartesian product of K_2 and K_3.

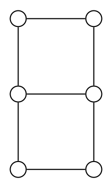

Figure 3: $K_2 \square K_3$

The last product we consider is the strong product of two graphs.

Definition 4. *The* ***strong product*** *of graphs G and H, denoted $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is*

$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

Figure 4 depicts the strong product of K_2 and K_3.

2 Previous Results

In this section, we state various results that were useful in our research. We give a short proof of any result that will be used in Section 3. The following theorem shows the monotonicity of t-tone colorings.

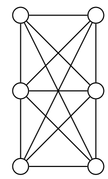

Figure 4: $K_2 \boxtimes K_3$

Theorem 5. *[4] If H is a subgraph of G, then $\tau_t(H) \leq \tau_t(G)$.*

Proof. Let f be a proper t-tone $\tau_t(G)$-coloring of G. The restriction of f to H must then be a proper t-tone coloring of H. Thus, $\tau_t(H)$ is no greater than $\tau_t(G)$. $\square$

In Section 3, we determine the 2-tone chromatic number of $K_m \times K_n$ for any $m, n \in \mathbb{Z}_{\geq 2}$. We shall use the following result to do so.

Theorem 6. *[4] If $G = K_n$ for any positive integer n, then $\tau_t(K_n) = tn$.*

Proof. Notice that each pair of distinct vertices of $V(K_n)$ is adjacent. Therefore, the set of t-element labels of the n vertices of K_n must be pairwise disjoint, and we have $\tau_t(K_n) = tn$. $\square$

Recall that a *k-partite graph* is a graph whose vertices can be partitioned into k disjoint sets, called *partite sets*, such that no two vertices within the same partite set are adjacent. A *complete k-partite graph*, denoted $K_{a_1,\cdots,a_k}$, is a k-partite graph with the additional property that any two distinct vertices contained in different partite sets are adjacent, and whose partite sets are of orders $a_1, \cdots, a_k$.

Theorem 7. *[4] Let $G = K_{a_1,a_2,\ldots,a_k}$ denote the complete k-partite graph, with partite sets of orders $a_1, a_2, \ldots, a_k$. Then*

$$\tau_2(G) = \sum_{i=1}^{k} \left\lceil \frac{1+\sqrt{8a_i+1}}{2} \right\rceil .$$

Proof. Let $G = K_{a_1,a_2,\ldots,a_k}$ and let $\chi(G) = k$. Let $A_1, \ldots, A_k$ denote the color classes of G. Let $|A_i| = a_i$, for $1 \leq i \leq k$. Now let f be a proper 2-tone $\tau_2(G)$-coloring of G. Notice that for any pair of distinct vertices $u, v \in A_i$, we have $d(u, v) = 2$. Hence, $|f(v) \cap f(u)| \leq 1$. So the number of colors required for the i^{th} partite set is s_i such that $\binom{s_i}{2} \geq a_i$. Thus, we get that $s_i = \left\lceil \frac{1+\sqrt{8a_i+1}}{2} \right\rceil$. Furthermore, since every element in A_i is adjacent to every element in A_j for all $i \neq j$ with $1 \leq i, j \leq k$, they cannot have any colors in common. Therefore, the 2-tone chromatic number of G is the sum of all such s_i, as desired. $\square$

There is a natural relationship between t-tone colorings and another variation of graph coloring known as a distance (d, k)-coloring.

Definition 8. *A* **distance** (d,k)**-coloring** *of* G *is a mapping* $f : V(G) \to [k]$ *such that if* u *and* v *are two distinct vertices of* $V(G)$ *where* $d(u,v) \leq d$*, then* $f(u) \neq f(v)$*. We use* $\chi_d(G)$ *to denote the minimum number* k *such that* G *has a proper distance* (d,k)*-coloring.*

As stated throughout the literature, determining $\chi_d(G)$ is equivalent to determining the chromatic number of the d^{th} power of a graph, defined below.

Definition 9. *Given a graph* G*, the* d^{th} **power of** G*, denoted* G^d*,is defined to be the graph with* $V(G^d) = V(G)$*, and for any two distinct vertices* $u, v \in V(G^d)$*,* $uv \in E(G^d)$ *if and only if* $d_G(u,v) \leq d$*.*

Since the focus of this paper is on 2-tone colorings, we will restrict ourselves to distance $(2,k)$-colorings. We now prove equality between $\chi_2(G)$ and $\chi(G^2)$.

Theorem 10. *For any graph* G*,* $\chi_2(G) = \chi(G^2)$*.*

Proof. Assume that $f : V(G) \to \{1, ..., \chi_2(G)\}$ is a proper distance $(2, \chi_2(G))$-coloring of G. If $uv \in E(G^2)$, then $d_G(u,v) \leq 2$ so $f(u) \neq f(v)$. Thus, f is a proper $\chi_2(G)$-coloring of G^2, and we have $\chi(G^2) \leq \chi_2(G)$. In the other direction, assume that $g : V(G^2) \to \{1, \cdots, \chi(G^2)\}$ is a proper $\chi(G^2)$-coloring of G^2. If $d_G(u,v) \leq 2$, then $uv \in E(G^2)$ so $g(u) \neq g(v)$. Thus, g is a distance $(2, \chi(G^2))$-coloring of G, and the result follows. □

As noted in Fonger, Phillips and Segroves [4], $\tau_2(G)$ is related to both $\chi(G)$ and $\chi(G^2)$ in the following way.

Theorem 11. *[4] For any graph* G*,* $\tau_2(G) \leq \chi(G) + \chi(G^2)$*.*

Proof. Let f_1 be a chromatic coloring of G using the colors $1, 2, ..., \chi(G)$, and f_2 a chromatic coloring of G^2 using the colors $\chi(G) + 1, ..., \chi(G^2)$. Then we define a 2-tone coloring f of G by setting $f(v) = \{f_1(v), f_2(v)\}$. Then, if $u,\ v$ are adjacent in G, we have that $f_1(v),\ f_2(v),\ f_1(u),\ and\ f_2(u)$ are four different colors. Similarly, if $d(u,v) = 2$ for vertices u and v of G, then $f_2(u) \neq f_2(v)$, and so the labels on u and v assigned by f are not the same. Therefore, f is a proper 2-tone coloring of G using $\chi(G) + \chi(G^2)$ colors. □

Bickle and Phillips give an upper bound for $\tau_2(G)$ in [2] based on $\Delta(G)$.

Theorem 12. *[2] For any graph* G*,* $\tau_2(G) \leq [\Delta(G)]^2 + \Delta(G)$*.*

This bound was improved upon by Cranston, Kim and Kinnersley in [3]. We will reference this result throughout Section 3 as a comparison to our results.

Theorem 13. *[3] For any graph* G*,* $\tau_2(G) \leq \lceil (2 + \sqrt{2})\Delta(G) \rceil$*.*

3 New Results

3.1 Direct Product

In this section, we focus on the direct product of two graphs. We restate the definition of the direct product for ease of reference.

Definition 14. *Given two graphs G and H, the **direct product** of G and H, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is*

$$E(G \times H) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(G) \text{ and } y_1y_2 \in E(H)\}.$$

Throughout this section, when we consider the direct product of two graphs G and H where $|G| = m$ and $|H| = n$, we will represent the vertices of G as $x_1, \cdots, x_m$ and the vertices of H as $y_1, \cdots, y_n$. Using this notation, for each $i \in [m]$ we define a *column* C_i as the set of all vertices with first coordinate x_i. Similarly, for each $j \in [n]$ we define the *row* R_j as the set of all vertices with second coordinate y_j . In particular, for $i \in [m]$, the i^{th} column is $C_i = \{(i, j) \mid j \in [n]\}$. Similarly, for $j \in [n]$ the j^{th} row is the set $R_j = \{(i, j) \mid i \in [m]\}$.

In order to find an upper and lower bound of the 2-tone chromatic number of the direct product of any two graphs G and H, we first considered the direct product of two complete graphs. The following property follows from the definition of the direct product.

Property 15. *For any $m, n \in \mathbb{Z}_{\geq 2}$*

$$V(K_m \times K_n) = \{(x_i, y_k) \mid \ 1 \leq i \leq m \text{ and } 1 \leq k \leq n\},$$

and

$$E(K_m \times K_n) = \{(x_i, y_k)(x_j, y_\ell) \mid \ i \neq j \text{ and } k \neq \ell\}.$$

Note the following consequence of Property 15.

Proposition 16. *Let m and n be positive integers such that $m \geq 2$ and $n \geq 3$. For any two distinct vertices (x_i, y_j) and (x_i, y_k) of $V(K_m \times K_n)$, $d((x_i, y_j), (x_i, y_k)) = 2$.*

Proof. Let $u, v \in V(K_m \times K_n)$ be two distinct vertices, and write $u = (x_i, y_j)$ and $v = (x_i, y_k)$ for some $1 \leq i \leq m$ and $1 \leq j < k \leq n$. Note that u and v are both contained in column C_i. Since $m \geq 3$, there exists $a \in \{1, \cdots, m\}$ such that $a \neq i$. Thus, $x_ix_a \in E(K_m)$. Moreover, since $n \geq 3$, there exists $b \in \{1, \cdots, n\}$ such that $b \neq j$ and $b \neq k$. Therefore, $y_jy_b \in E(H)$ and $y_by_k \in E(H)$. So $(x_i, y_j)(x_a, y_b)(x_i, y_k)$ is a path in $K_m \times K_n$ of length 2. It follows that $d(u, v) = 2$. $\square$

Recall from Section 1 that given a graph G and a t-tone k-coloring f of G, we call $f(v)$ the *label of* v and the elements of $[k]$ *colors*. Additionally, for any set of vertices $A \subseteq V(G)$, we define the *set of colors contained in the labels associated with* A to be

$$c(A) = \{c \in [k] \mid \ c \in f(v) \text{ for some } v \in A\}.$$

Theorem 17. *If $m, n \in \mathbb{N}$ where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, then*

$$\tau_2(K_m \times K_n) \geq \min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\}.$$

Proof. First consider the case where $m = n = 2$. Since $K_2 \times K_2 \cong 2K_2$,

$$\tau_2(K_2 \times K_2) = 2\tau_2(K_2) = 4.$$

One can easily verify that this is the minimum of the three functions.

Now consider all other cases where $m \geq 2, n \geq m$ and $n \neq 2$. Let f be a minimum 2-tone k-coloring of $K_m \times K_n$. For each $1 \leq i \leq m$, define A_i as the set of all vertices $v \in C_i$ such that for each $a \in f(v)$, there exists a vertex $w \in C_i$ with $w \neq v$ and $a \in f(v) \cap f(w)$. Note the following consequence of this partition of each C_i. Fix $i \in \{1, \cdots, m\}$ and let v and w be distinct vertices of A_i as described above. That is, for some $a \in f(v)$ we have $a \in f(v) \cap f(w)$. This implies that for all $j \in \{1, \cdots, m\}$ such that $j \neq i$ and for any $u \in A_j$, $a \notin f(u)$ since u is adjacent to at least one of v or w by Property 15. Therefore, the set of colors contained in the labels associated with A_i is disjoint from the set of colors contained in the labels associated with A_j. That is, $c(A_i) \cap c(A_j) = \emptyset$.

For each $1 \leq i \leq m$, let $s_i = |c(A_i)|$. We may assume that $\min_{1 \leq i \leq m} s_i = s_\ell$ for some $\ell \in \{1, \cdots, m\}$. Thus, the number of distinct colors contained in the labels associated with $\cup_{i=1}^m A_i$ is at least ms_ℓ. Furthermore, for each $(x_\ell, y_j) \in C_\ell \backslash A_\ell$, there exists a color $a \in f((x_\ell, y_j))$ such that a is not contained in any other label associated with C_ℓ. So for each $1 \leq i \leq m$, $a \notin c(A_i)$ by definition of A_i. Since f is a proper 2-tone k-coloring of $K_m \times K_n$, we know that $k \geq ms_\ell + |C_\ell \backslash A_\ell|$. We now determine the minimum k based on the value of $|C_\ell \backslash A_\ell|$.

Case 1: Assume that $|C_\ell \backslash A_\ell| = n$. Thus, $A_\ell = \emptyset$ and $s_\ell = 0$. Let j represent the number of columns such that $s_i > 0$ for some $i \in \{1, \cdots, m\}$. Since $s_\ell = 0$, $m - j \geq 1$ or equivalently $m \geq j + 1$. Let $A = \{C_i \mid i \in \{1, \cdots, m\}$ and $s_i = 0\}$. For indexing purposes, we shall write $A = \{C_{\alpha(1)}, \cdots, C_{\alpha(m-j)}\}$ where $\alpha(i) \in \{1, \cdots, m\}$ for $1 \leq i \leq m - j$. Since $m - j \leq m \leq n$, there exists a set $W = \{v_{\alpha(1)}, \cdots, v_{\alpha(m-j)}\}$ such that $v_{\alpha(i)} \in C_{\alpha(i)}$ for each $\alpha(i)$ and if $\alpha(i) \neq \alpha(j)$, then $v_{\alpha(i)}$ and $v_{\alpha(j)}$ are contained in different rows. This implies that the induced subgraph of W is a clique, and it follows that $|f(v_{\alpha(i)}) \cap f(v_{\alpha(j)})| = 0$ when $\alpha(i) \neq \alpha(j)$. Furthermore, the $n - (m - j)$ remaining vertices $(x_\ell, y_j) \in C_\ell \backslash A_\ell$ where $R_j \cap W = \emptyset$ contain at least $n - (m - j)$ colors that are not contained in $c(W)$. Thus, $c(B) \geq 2(m - j) + n - (m - j)$.

Finally, for each column C_i where $s_i > 0$, we know that $s_i \geq 2$. Thus,

$$\begin{aligned} k &\geq 2(m-j) + n - (m-j) + 2j \\ &= m + n + j \\ &\geq m + n. \end{aligned}$$

Case 2: Assume that $|C_\ell \backslash A_\ell| = 0$. It follows that $A_\ell = C_\ell$ and $|A_\ell| = n$. Furthermore, since s_ℓ represents the number of distinct colors contained in the labels associated with A_ℓ, we know that $s_\ell \geq 2$. Using the same argument as in Theorem 7 and the fact that any two distinct vertices $u, v \in A_\ell$ satisfy $d(u,v) = 2$, we know that $\binom{s_\ell}{2} \geq n$. Using the quadratic formula, this implies that $s_\ell \geq \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. Consequently, $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

Case 3: Assume that $n > |C_\ell \backslash A_\ell| > 0$. If $\binom{s_\ell}{2} > n$, then clearly $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. So assume $\binom{s_\ell}{2} \leq n$, or equivalently $2 \leq s_\ell \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. We have $n = |C_\ell| = |A_\ell| + |C_\ell \backslash A_\ell|$, which implies $|C_\ell \backslash A_\ell| = n - |A_\ell|$. As in Case 1, we know $\binom{s_\ell}{2} \geq |A_\ell|$. Thus,

$$\begin{aligned} n - \binom{s_\ell}{2} &\leq n - |A_\ell| \\ \Longrightarrow n - \binom{s_\ell}{2} &\leq |C_\ell \backslash A_\ell|. \end{aligned}$$

Therefore,

$$ms_\ell + |C_\ell \backslash A_\ell| \geq ms_\ell + n - \binom{s_\ell}{2} = ms_\ell + n - \frac{s_\ell(s_\ell - 1)}{2}.$$

So we consider the function $g(s) = ms + n - \frac{s(s-1)}{2}$ over the interval $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. One can easily verify that $g'(s) = m - s + \frac{1}{2}$ and $g''(s) = -1$. Thus, g is concave down for all values of $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$, and over this interval g has a local maximum when $s = m + \frac{1}{2}$. Therefore, the local minimums for g occur when $s = 2$ and $s = \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. Letting $t = \frac{1+\sqrt{1+8n}}{2}$, it follows that

$$\begin{aligned} k &\geq ms_\ell + |C_\ell \backslash A_\ell| \\ &\geq \min_s \quad ms + n - \frac{s(s-1)}{2} \\ &\quad\ \text{s.t.} \quad 2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor \\ &\geq \min\left\{2m + n - 1, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}\right\}. \end{aligned}$$

Since $2m + n - 1 > m + n$ and f is assumed to be a minimum 2-tone coloring, we shall take $\min\left\{2m + n - 1, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}\right\} = m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}$.

Note that these cases sometimes overlap. For example, $\binom{s}{2} = n$ implies that $\lfloor t \rfloor = \lceil t \rceil = t$ and $n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} = 0$, resulting in $m\lceil t \rceil = m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}$. In any case, we have

$$\tau_2(K_m \times K_n) \geq \min\left\{m\lceil t \rceil, m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}\right\}.$$

$\square$

Theorem 18. *If $m, n \in \mathbb{N}$ where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, then*

$$\tau_2(K_m \times K_n) = \min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\}.$$

Proof. From Theorem 17, all that remains to be shown is that

$$\tau_2(K_m \times K_n) \geq \min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\}.$$

Given $m, n \in \mathbb{N}$ where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, we shall construct different 2-tone colorings which depend on the value of $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\}$.

Case 1: First assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m\lceil t\rceil$. Choose m pairwise disjoint sets each containing $\lceil t\rceil$ distinct colors, and denote each set of colors S_i for $1 \leq i \leq m$. Since $\binom{\lceil t\rceil}{2} \geq n$, for each $1 \leq i \leq m$ there exist n distinct combinations containing two colors from the set S_i. Thus, we may define $f : V(K_m \times K_n) \to \binom{[\lceil t\rceil]}{2}$ to be any mapping such that for each $1 \leq i \leq m$ the restriction of f to the set of vertices in C_i is an injective mapping to the set of combinations containing two colors from the set S_i. Figure 5 illustrates this particular 2-tone coloring for $K_2 \times K_6$. To see that f is a proper 2-tone coloring of $K_m \times K_n$, let u and v be distinct vertices of $V(K_m \times K_n)$. If u and v are not contained in the same column, then $f(u) \cap f(v) = \emptyset$. So assume $u, v \in C_i$ for some $i \in \{1, \cdots, m\}$. Since $d(u, v) = 2$, we must show that $|f(u) \cap f(v)| \leq 1$. However, this follows from the fact that f does not assign any label to more than one vertex of C_i. Therefore, f is a proper 2-tone coloring.

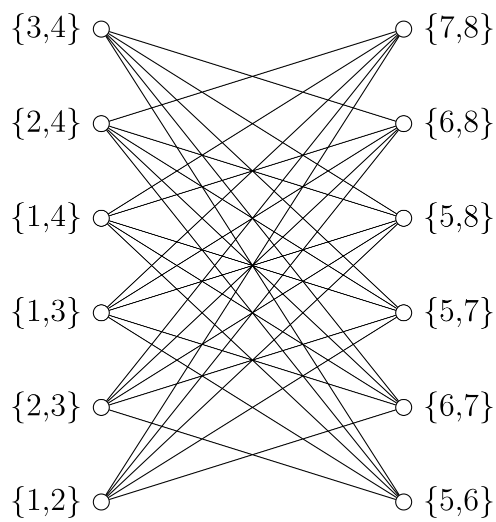

Figure 5: $K_2 \times K_6$

Case 2: Next, assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m+n$. Let f_1 be a proper coloring of K_m and f_2 be a proper coloring of K_n defined as follows:

$$\begin{aligned} f_1 : V(K_m) &\to [m] \\ x_i &\mapsto i \\ f_2 : V(K_n) &\to \{m+1, \cdots, m+n\} \\ y_k &\mapsto k+m. \end{aligned}$$

Define the following function on $V(K_m \times K_n)$:

$$\begin{aligned} g : V(K_m \times K_n) &\to \binom{[m+n]}{2} \\ (x_i, y_k) &\mapsto \{f_1(x_i), f_2(y_k)\}. \end{aligned}$$

Figure 6 illustrates a particular 2-tone coloring of this type for $K_2 \times K_3$.

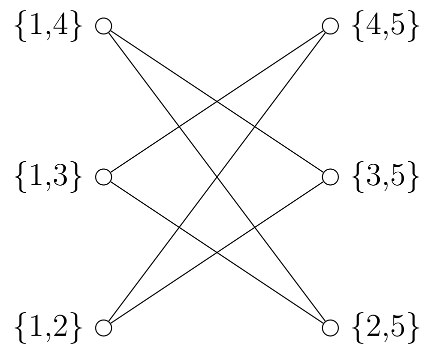

Figure 6: $K_2 \times K_3$

We claim g is a proper 2-tone coloring of $K_m \times K_n$. Clearly $|g((x,y))| = 2$ for all $(x,y) \in V(K_m \times K_n)$. Let (x_i, y_k) and (x_j, y_ℓ) be two distinct vertices of $V(K_m \times K_n)$ where $1 \leq i,j \leq m$ and $1 \leq k,\ell \leq n$. Then $g((x_i,y_k)) = \{i, k+m\}$ and $g((x_j,y_\ell)) = \{j, \ell+m\}$. If (x_i,y_k) and (x_j,y_ℓ) are adjacent, then $i \neq j$ and $k \neq \ell$. Thus, $|g((x_i,y_k)) \cap g((x_j,y_\ell))| = 0$. If $d((x_i,y_k),(x_j,y_\ell)) = 2$, then either $i \neq j$ or $k \neq \ell$. In any case, $|g((x_i,y_k)) \cap g((x_j,y_\ell))| \leq 1$. Therefore, g is a proper 2-tone coloring of $K_m \times K_n$, and we may conclude that $\tau_2(K_m \times K_n) \leq m+n$.

Case 3: Assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}$. Note that $t = \frac{1+\sqrt{1+8n}}{2}$ is the only positive solution to $\binom{t}{2} = n$. Therefore, $\lfloor t\rfloor$ satisfies $\binom{\lfloor t\rfloor}{2} \leq n$. Let $s = \binom{\lfloor t\rfloor}{2}$ and consider the subgraph H of $K_m \times K_n$ induced by the set $\{(x_i, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq s\}$. Thus, $H \cong K_m \times K_s$. As in Case 2, choose m pairwise disjoint sets of $\lfloor t\rfloor$ distinct colors and denote each set S_i for $1 \leq i \leq m$. Define $f_1 : V(H) \to \binom{[\lfloor t\rfloor]}{2}$ to be any mapping such that for each $1 \leq i \leq m$, the restriction of f_1 to the set of vertices of C_i is an injective mapping to the set of combinations containing two colors from the set S_i. A similar argument as in Case 2 can be used to show that f_1 is a proper 2-tone coloring of H.

Next, choose $n-s$ distinct colors each of which are not contained in the set $\cup_{i=1}^{m} S_i$, and label these colors $\{t_{s+1}, \cdots, t_n\}$. Additionally, for each $i \in \{1, \cdots, m\}$, choose one color from the set S_i and call it c_i.

Notice that $V(K_m \times K_n)\backslash V(H) = \{(x_i, y_j) \mid \ 1 \leq i \leq m, s+1 \leq j \leq n\}$. Define

$$f_2 : V(K_m \times K_n)\backslash V(H) \to \binom{[m+n-s]}{2}$$
$$(x_i, y_j) \mapsto \{c_i, t_j\}.$$

We claim that f_2 is a proper 2-tone coloring of $(K_m \times K_n)\backslash H$. To see this, let (x_i, y_k) and (x_j, y_ℓ) be two distinct vertices of $V(K_m \times K_n)\backslash V(H)$ for some $1 \leq i, j \leq m$ and $s+1 \leq k, \ell \leq n$. If (x_i, y_k) and (x_j, y_ℓ) are adjacent, then $i \neq j$ and $k \neq \ell$. Since S_i and S_j are two disjoint sets of colors, we know that $c_i \neq c_j$. Moreover, we know that $t_k \neq t_\ell$ since $k \neq \ell$. Thus, $|f_2((x_i, y_k)) \cap f_2((x_j, y_\ell))| = 0$. If $d((x_i, y_k), (x_j, y_\ell)) = 2$, then either $i \neq j$ or $k \neq \ell$. It follows that $|f_2((x_i, y_k)) \cap f_2((x_j, y_\ell))| \leq 1$. Therefore, f_2 is a proper 2-tone coloring of $(K_m \times K_n)\backslash H$.

Now define $g : V(K_m \times K_n) \to \binom{[m\lfloor t\rfloor + n - s]}{2}$ such that

$$g(u) = \begin{cases} f_1(u) & \text{if } u \in V(H) \\ f_2(u) & \text{otherwise.} \end{cases}$$

Figure 7 illustrates a particular 2-tone coloring of this type for $K_2 \times K_{11}$.

To see that g is a proper 2-tone coloring, we only need to consider when $u \in V(H)$ and $v \notin V(H)$. Write $u = (x_i, y_k)$ and $v = (x_j, y_\ell)$ for some $i, j \in \{1, \cdots, m\}$, $k \in \{1, \cdots, s\}$, and $\ell \in \{s+1, \cdots, n\}$. By definition $g(v) = \{c_j, t_\ell\}$, and we know $t_\ell \notin g(u)$ since $u \in V(H)$. So if u and v are located in the same column, then $|g(u) \cap g(v)| \leq 1$. If u and v are not located in the same column, then $i \neq j$ and $c_j \notin g(u)$ since $c_j \notin S_i$. It follows that $|g(u) \cap g(v)| = 0$. Therefore, g is a proper 2-tone coloring of $K_m \times K_n$ using $m\lfloor t\rfloor + n - \binom{\lfloor t\rfloor}{2}$ colors.

□

Using similar ideas found in Theorems 17 - 18, we can bound the value of $\tau_2(G \times H)$ given any graphs G and H. We make use of the following general lower bound given in [4].

Theorem 19. *[4] Let G be a graph and let $\Delta(G) = d$. Then*

$$\tau_2(G) \geq \left\lceil \frac{\sqrt{8d+1}+5}{2} \right\rceil.$$

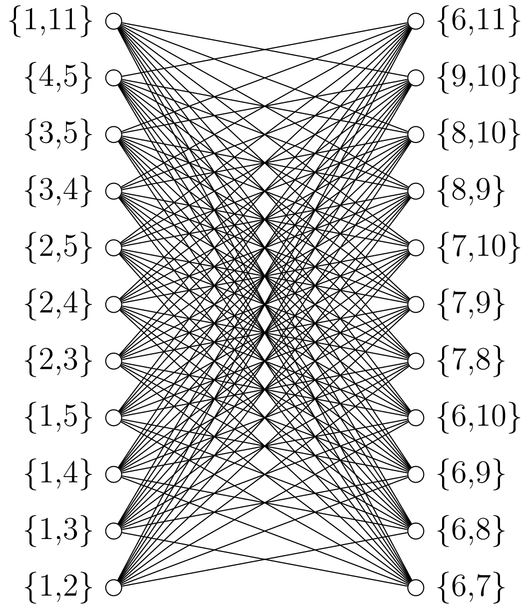

Figure 7: $K_2 \times K_{11}$

Theorem 20. *Given two graphs G and H,*

$$\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)})\right\} \leq \tau_2(G \times H) \leq \chi(G^2) + \chi(H^2).$$

Proof. We first show that for any graphs G and H, we have $\tau_2(G \times H) \leq \chi(G^2) + \chi(H^2)$. Assume $\chi_2(G) = k_1$ and $\chi_2(H) = k_2$. Let $f_1 : V(G) \to \{1, ..., k_1\}$ be a distance-2 coloring of G, and let $f_2 : V(H) \to \{k_1 + 1, ..., k_1 + k_2\}$ be a distance-2 coloring of H. Define

$$g : V(G \times H) \to \binom{[k_1 + k_2]}{2}$$

such that

$$(x, y) \mapsto \{f_1(x), f_2(y)\} \qquad \text{for all } x \in V(G) \text{ and } y \in V(H).$$

We claim that g is a proper 2-tone coloring of $G \times H$. Clearly $|g((x, y))| = 2$ for all $(x, y) \in V(G \times H)$. Let (u, v) and (w, z) be two distinct vertices of $V(G \times H)$. If (u, v) and (w, z) are adjacent, then $uw \in E(G)$ and $vz \in E(H)$. It follows that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. Since the range of f_1 is disjoint from the range of f_2 as mappings, we have $|g((u, v)) \cap g((w, z))| = 0$.

If $d_{G\times H}((u, v), (w, z)) = 2$, then there exists a vertex $(x, y) \in V(G \times H)$ such that uxw is a path in G and vyz is a path in H. Since $d_G(u, w) \leq 2$ and $d_H(v, z) \leq 2$, we know that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. Thus, $|g((u, v)) \cap g((w, z))| = 0$, and we may conclude that g is a proper 2-tone coloring of $G \times H$.

Next, we show that $\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)})\right\} \leq \tau_2(G \times H)$. We shall assume that $\omega(G)$ is the clique number of G, and $\omega(H)$ is the clique number of H. By definition of the direct product, $K_{\omega(G)} \times K_{\omega(H)}$ is a subgraph of $G \times H$. Thus, $\tau_2(K_{\omega(G)} \times K_{\omega(H)}) \leq \tau_2(G \times H)$. On the other hand, we know $\Delta(G \times H) = \Delta(G)\Delta(H)$. So by Theorem 19, we know that $\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil \leq \tau_2(G \times H)$. Therefore,

$$\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)})\right\} \leq \tau_2(G \times H).$$

□

It should be noted that there exist graphs G and H such that the upper bound in Theorem 20 is better than applying Theorem 13. For example, consider the graph $P_3 \times P_4$ in Figure 8. One can easily verify that the labeling shown in Figure 8 is in fact a 2-tone coloring. Thus, $\tau_2(P_3 \times P_4) \leq 5$ which is an improvement from the bound given in Theorem 13 of

$$\begin{aligned}
\tau_2(P_3 \times P_4) &\leq \left\lceil(2+\sqrt(2))\Delta(P_3 \times P_4)\right\rceil \\
&= \left\lceil(2+\sqrt{2})4\right\rceil \\
&= 14.
\end{aligned}$$

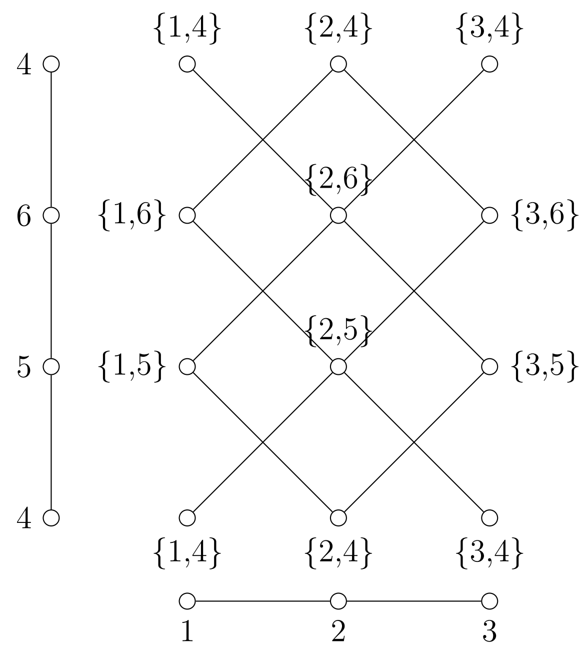

Figure 8: A 2-tone coloring of $P_3 \times P_4$

3.2 Cartesian Product

We now focus on the Cartesian product of two graphs. We restate the definition of this product below for ease of reference.

Definition 21. *The **Cartesian product** of graphs G and H, denoted $G\Box H$, is the graph whose vertex set is $V(G) \times V(H)$ with edge set*

$$E(G\Box H) = \{(x_1, y_1)(x_2, y_2)|\ x_1y_1 \in E(G) \text{ and } x_2 = y_2, \text{ or } x_1 = y_1 \text{ and } x_2y_2 \in E(H)\}.$$

In this particular product, we have an obvious lower bound for the 2-tone chromatic number of $G\Box H$.

Theorem 22. *Given two graphs G and H,*

$$\max\{\tau_2(G), \tau_2(H)\} \leq \tau_2(G\Box H).$$

Proof. This follows from the fact that G and H are both subgraphs of $G\Box H$. □

In terms of an upper bound, it is stated in [1] that $\tau_2(G\Box H) \leq \tau_2(G)\tau_2(H)$, but that this bound could be improved. We give an upper bound for $\tau_2(G\Box H)$ in terms of $\max\{\chi(G^2), \chi(H^2)\}$ depending on the parity of this value.

Theorem 23. *Given two graphs G and H where $\max\{\chi(G^2), \chi(H^2)\} = \chi(G^2)$,*

$$\tau_2(G \ \Box \ H) \leq \begin{cases} 2\chi(G^2) & \text{if } \chi(G^2) \text{ is odd} \\ 2(\chi(G^2) + 1) & \text{otherwise.} \end{cases}$$

Proof. Assume that $\max\{\chi(G^2), \chi(H^2)\} = \chi(G^2)$. If $\chi(G^2)$ is an even integer, then we will let $k = \chi(G^2) + 1$. Otherwise, we will let $k = \chi(G^2)$. Let $f_1 : V(G) \mapsto [k]$ be a proper distance $(2, k)$-coloring of G, and let $f_2 : V(H) \mapsto [k]$ be a proper distance $(2, k)$-coloring of H.

Define $g : V(G\Box H) \mapsto \binom{[2k]}{2}$ such that

$$(x, y) \mapsto \{f_1(x) + f_2(y) \pmod k, (f_2(y) - f_1(x) \pmod k) + k\}.$$

We will first show that g assigns two distinct colors to each vertex of $G\Box H$. Let $(x, y) \in V(G\Box H)$ and write $g((x, y)) = \{a, b\}$. Since $a = f_1(x) + f_2(y) \pmod k$, it follows that $a \in [k]$. On the other hand, $b = (f_2(y) - f_1(x) \pmod k) + k$ so $b \in \{k + 1, \cdots, 2k\}$. Thus, $|g((x, y))| = 2$.

Next, we show that g satisfies the distance criteria for 2-tone colorings. Let (u, v) and (w, z) be two distinct vertices of $V(G\Box H)$.

Case 1: Suppose that $d_{G\Box H}((u,v)(w,z)) = 1$. Then either $u = w$ and $d_H(v,z) = 1$ or $v = z$ and $d_G(u,w) = 1$. If $u = w$ and $d_H(v,z) = 1$, then we know that $f_1(u) = f_1(w)$ and $f_2(v) \neq f_2(z)$. This implies that

$$f_1(u) + f_2(v) \not\equiv f_1(w) + f_2(z) \pmod{k}.$$

Moreover,

$$f_2(v) - f_1(u) \not\equiv f_2(z) - f_1(w) \pmod{k}$$
$$\implies (f_2(v) - f_1(u) \pmod{k}) + k \neq (f_2(z) - f_1(w) \pmod{k}) + k.$$

So $|g((u,v)) \cap g((w,z))| = 0$.

A similar argument shows that $|g((u,v)) \cap g((w,z))| = 0$ if $v = z$ and $d_G(u,w) = 1$.

Case 2: Suppose that $d_{G\Box H}((u,v)(w,z)) = 2$. Then exactly one of the following will be true.

$$\begin{array}{ll} (a) & u = w \text{ and } d_H(v,z) = 2, \text{ or} \\ (b) & v = z \text{ and } d_G(u,w) = 2, \text{ or} \\ (c) & d_G(u,w) = 1 \text{ and } d_H(v,z) = 1. \end{array}$$

In the case of either (a) or (b), a similar argument as in Case 1 shows $|g((u,v)) \cap g((w,z))| = 0$. So assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. If $|g((u,v)) \cap g((w,z))| \leq 1$, we are done. So suppose that $g((u,v)) = g((w,z))$. Thus,

$$\begin{aligned} (f_2(v) - f_1(u) \pmod{k}) + k &= (f_2(z) - f_1(w) \pmod{k}) + k \\ \implies f_2(v) - f_1(u) &\equiv f_2(z) - f_1(w) \pmod{k} \\ \implies f_2(v) - f_2(z) &\equiv f_1(u) - f_1(w) \pmod{k}. \end{aligned} \tag{1}$$

Moreover,

$$\begin{aligned} f_1(u) + f_2(v) &\equiv f_1(w) + f_2(z) \pmod{k} \\ \implies f_1(u) - f_1(w) &\equiv f_2(z) - f_2(v) \pmod{k} \\ \implies f_2(v) - f_1(z) &\equiv f_2(z) - f_2(v) \pmod{k} \qquad \text{from (1)} \\ \implies 2f_2(v) &\equiv 2f_2(z) \pmod{k}. \end{aligned}$$

However, this cannot happen since $f_2(v) \not\equiv f_2(z) \pmod{k}$ and $\gcd(2,k) = 1$. Thus, $|g((u,v)) \cap g((w,z))| \leq 1$.

□

Although the upper bound in Theorem 23 does not involve $\tau_2(G)$ or $\tau_2(H)$, it should be noted that there are graphs for which the upper bound is best possible. For example, consider the graph $P_3 \Box P_3$. We know $\tau_2(P_3) = 5$ and $\chi(P_3^2) = 3$. Moreover, $P_3 \Box P_3$ contains a cycle of length 4 and since $\tau_2(C_4) = 6$, it must be the case that $6 \leq \tau_2(P_3 \Box P_3)$. Figure 9 shows a proper 2-tone coloring using only 6 colors. Thus, $\tau_2(P_3 \Box P_3) = 2\chi(G^2) = 6$.

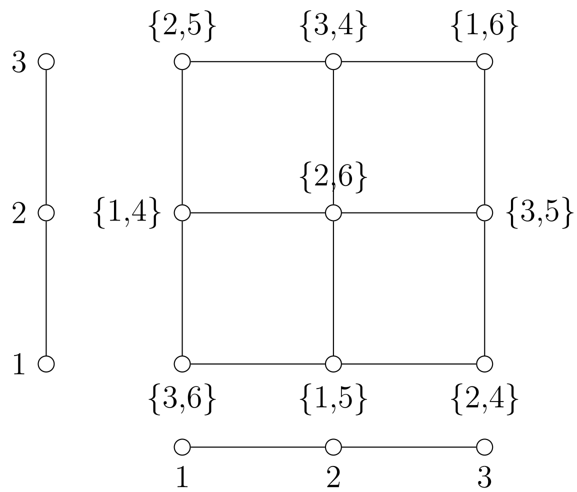

Figure 9: A 2-tone coloring of $P_3 \square P_3$

3.3 Strong Product

The last graph product that we consider is the strong product $G \boxtimes H$. We restate the definition of this product below for ease of reference.

Definition 24. *The* ***strong product*** *of graphs G and H, denoted $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is*

$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

Using similar ideas to those found in Sections 3.1 and 3.2, we have the following upper and lower bounds for $\tau_2(G \boxtimes H)$.

Theorem 25. *Given two graphs G and H,*

$$\max\{\tau_2(G \times H), \tau_2(G \square H)\} \leq \tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}.$$

Proof. Note that $G \square H$ and $G \times H$ are both subgraphs of $G \boxtimes H$. Thus, $\max\{\tau_2(G \square H), \tau_2(G \times H)\} \leq \tau_2(G \boxtimes H)$ by Theorem 5.

Next, we will prove that $\tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}$. Without loss of generality, we may assume $\tau_2(G)\chi(H^2) \leq \chi(G^2)\tau_2(H)$. Let f_1 be a proper 2-tone coloring of G using the colors $\{1, 2, ...\tau_2(G)\}$. Let f_2 be a proper distance $(2, k)$-coloring of H using the colors $\{1, \tau_2(G) + 1, 2\tau_2(G) + 1, ..., (k-1)\tau_2(G) + 1\}$ where $k = \chi(H^2)$. Define $g : V(G \boxtimes H) \to \binom{[k\tau_2(G)+1]}{2}$ such that for each $(x, y) \in V(G \boxtimes H)$ and for each $c \in f_1(x)$, we have $c + f_2(y) \in g((x, y))$.

We show that g is a proper 2-tone coloring of $G \boxtimes H$. Let (u, v) and (w, z) be vertices of $V(G \boxtimes H)$.

Case 1: Assume that $d_{G\boxtimes H}((u,v),(w,z)) = 1$. By definition of the strong product, exactly one of the following will be true.

1) $d_G(u,w) = 1$ and $v = z$,
2) $u = w$ and $d_H(v,z) = 1$, or
3) $d_G(u,w) = 1$ and $d_H(v,z) = 1$.

We show that $|g((u,v)) \cap g((w,z))| = 0$ in each of the above cases.

1) Assume $d_G(u,w) = 1$ and $v = z$. Since f_1 is a proper 2-tone coloring of G, $f_1(u) \cap f_1(w) = \emptyset$. Thus, we can write $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_i \neq c_j$ for $1 \leq i < j \leq 4$. Since $f_2(v) = f_2(z)$, we know for $i \in \{1,2\}$ and $j \in \{3,4\}$ that $c_i + f_2(v) \neq c_j + f_2(z)$. Therefore, $|g((u,v)) \cap g((w,z))| = 0$.
2) Assume $u = w$ and $d_H(v,z) = 1$. Since f_2 is a proper distance $(2,k)$-coloring of H, $f_2(v) \neq f_2(z)$ and we may write $f_2(v) = i\tau_2(G) + 1$ and $f_2(z) = j\tau_2(G) + 1$ for some $0 \leq i < j \leq k-1$. Let $f_1(u) = \{c_1, c_2\}$ where $c_1 \neq c_2$. Thus,
$$g((u,v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$
and
$$g((w,z)) = \{c_1 + j\tau_2(G) + 1, c_2 + j\tau_2(G) + 1\}.$$
It is clear that $c_1 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1$ since $i < j$. Similarly, $c_2 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1$. Note that if $c_1 + i\tau_2(G) + 1 = c_2 + j\tau_2(G) + 1$, then
$$c_1 - c_2 = (j-1)\tau_2(G).$$
We know that $c_1 - c_2 \neq 0$ since $i < j$. On the other hand, $c_1 - c_2$ cannot be a multiple of $\tau_2(G)$ since $1 \leq c_1, c_2 \leq \tau_2(G)$. Therefore,
$$c_1 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1,$$
and a similar argument shows that
$$c_2 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1.$$
Thus, $|g((u,v)) \cap g((w,z))| = 0$.
3) Assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \cap f_1(w) = \emptyset$ and $f_2(v) \neq f_2(z)$. As before, let $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_a \neq c_b$ when $1 \leq a < b \leq 4$. Also, write $f_2(v) = i\tau_2(G) + 1$ and $f_2(z) = j\tau_2(G) + 1$ for some $0 \leq i < j \leq k-1$. Thus,
$$g((u,v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$
and
$$g((w,z)) = \{c_3 + j\tau_2(G) + 1, c_4 + j\tau_2(G) + 1\}.$$

Again, we see that $c_1+i\tau_2(G)+1 \neq c_2+i\tau_2(G)+1$ since $c_1 \neq c_2$. Similarly, $c_3+j\tau_2(G)+1 \neq c_4 + j\tau_2(G)+1$ since $c_3 \neq c_4$. Furthermore, for any $a \in \{1,2\}$ and $b \in \{3,4\}$, we know

$$c_a + i\tau_2(G) + 1 \neq c_b + j\tau_2(G) + 1$$

since $c_a - c_b$ cannot be a multiple of $\tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| = 0$.

Case 2: Assume that $d_{G\boxtimes H}((u,v),(w,z)) = 2$. Necessarily, $d_G(u,w) \leq 2$ and $d_H(v,z) \leq 2$. Thus, $|f_1(u) \cap f_1(w)| \leq 1$ so there exist $a \in f_1(u)$ and $b \in f_1(w)$ such that $a \neq b$. Furthermore, since $d_H(v,z) \leq 2$, we may assume there exist $0 \leq i < j \leq k-1$ such that $f_2(v) = i\tau_2(G)+1$ and $f_2(z) = j\tau_2(G)+1$. We have already seen that this implies $a + i\tau_2(G) + 1 \neq b + j\tau_2(G) + 1$ since $i < j$ and $1 \leq a, b \leq \tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| \leq 1$.

$\square$

Note that for $P_3 \boxtimes P_3$, we can find a 2-tone 6-coloring as shown in Figure 10. This coloring is best possible since $P_3 \boxtimes P_3$ contains C_4 and $\tau_2(C_4) = 6$. However, in this case Theorem 25 gives bounds of

$$5 = \max\{\tau_2(P_3 \Box P_3), \tau_2(P_3 \times P_3)\} \leq \tau_2(P_3 \boxtimes P_3) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 15.$$

This alone shows that perhaps a better upper bound exists in terms of other graph properties. On the other hand, $K_3 \boxtimes K_3 \equiv K_9$ so we know $\tau_2(K_3 \boxtimes K_3) = 18$ as seen in Figure 11. In this particular case, we have

$$6 = \min\{\tau_2(K_3 \Box K_3), \tau_2(K_3 \times K_3)\} \leq \tau_2(K_3 \boxtimes K_3) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 18,$$

which shows the upper bound in Theorem 25 is sharp.

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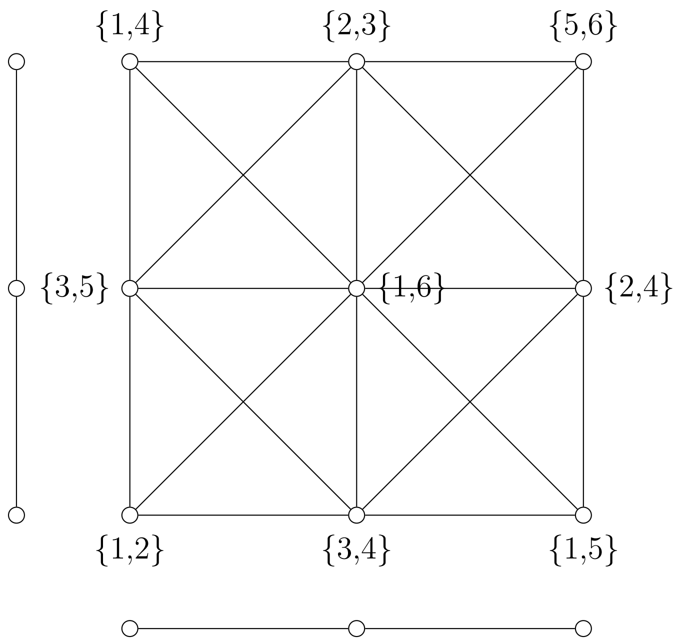

Figure 10: $P_3 \boxtimes P_3$

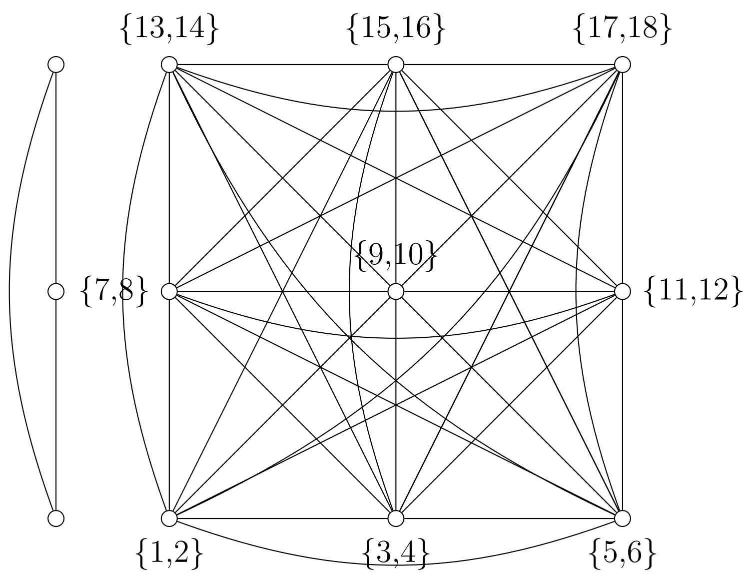

Figure 11: $K_3 \boxtimes K_3$