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## Some recent results in permutation decoding

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## Introduction

Permutation decoding was first developed by Jessie MacWilliams [9] in the early 60's. It can be used when a code has sufficiently many automorphisms to ensure the existence of a set of automorphisms called a PD-set:

**Definition 1** *A PD-set for a code is a set  $S$  of automorphisms of the code which is such that, if the code can correct  $t$  errors, then every possible error vector of weight  $t$  or less can be moved by some member of  $S$  out of the information positions.*

*More specifically, if  $\mathcal{I} = \{1, \dots, k\}$  are the information positions and  $\mathcal{C} = \{k+1, \dots, n\}$  the check positions, then every  $t$ -tuple from  $\{1, \dots, n\}$  can be moved by some element of  $S$  into  $\mathcal{C}$ .*

Such a set will fully use the error-correction potential of the code, but that such a set exists at all is clearly not always true, and existence is not invariant under code equivalence. There is a bound on the minimum size that the set  $S$  may have.

## Algorithm for permutation decoding

$C$  is a  $q$ -ary  $t$ -error-correcting  $[n, k, d]_q$  code;  $d = 2t + 1$  or  $2t + 2$ .

$k \times n$  generator matrix for  $C$ :  $G = [I_k | A]$ .

Any  $k$ -tuple  $v$  is encoded as  $vG$ . The first  $k$  columns are the information symbols, the last  $n - k$  are check symbols.

$(n - k) \times n$  check matrix for  $C$ :  $H = [-A^T | I_{n-k}]$ .

Suppose  $x$  is sent and  $y$  is received and at most  $t$  errors occur.

$\mathcal{S} = \{g_1, \dots, g_s\}$  is a PD-set for  $C$ .

- For  $i = 1, \dots, s$ , compute  $yg_i$  and the syndrome  $s_i = H(yg_i)^T$  until an  $i$  is found such that the weight of  $s_i$  is  $t$  or less;
- if  $u = u_1u_2 \dots u_k$  are the information symbols of  $yg_i$ , compute the codeword  $c = uG$ ;
- decode  $y$  as  $cg_i^{-1}$ .

## Why permutation decoding works

**Result 1** Let  $C$  be an  $[n, k, d]_q$   $t$ -error-correcting code. Suppose  $H$  is a check matrix for  $C$  in standard form, i.e. such that  $I_{n-k}$  is in the redundancy positions. Let  $y = c + e$  be a vector, where  $c \in C$  and  $e$  has weight  $\leq t$ . Then the information symbols in  $y$  are correct if and only if the weight of the syndrome  $Hy^T$  of  $y$  is  $\leq t$ .

## Minimum size for a PD-set

Counting shows that there is a minimum size a PD-set can have; most the sets known have size larger than this minimum. The following is due to Gordon [5], using a result of Schönheim [11]:

**Result 2** *If  $\mathcal{S}$  is a PD-set for a  $t$ -error-correcting  $[n, k, d]_q$  code  $C$ , and  $r = n - k$ , then*

$$|\mathcal{S}| \geq \left[ \frac{n}{r} \left[ \frac{n-1}{r-1} \left[ \cdots \left[ \frac{n-t+1}{r-t+1} \right] \cdots \right] \right] \right].$$

(Proof in Huffman [6].)

**Example:** The binary extended Golay code, parameters  $[24, 12, 8]$ , has  $n = 24$ ,  $r = 12$  and  $t = 3$ , so

$$|\mathcal{S}| \geq \left[ \frac{24}{12} \left[ \frac{23}{11} \left[ \frac{22}{10} \right] \right] \right] = 14$$

and PD-sets of this size has been found (see Gordon [5] and Wolfmann [12]).

## Magma results

Some computational examples using Magma [3]:

1. for  $C$  the  $[28, 21, 4]_2$  code of the hermitian unitary  $2-(28,4,1)$  has  $|S| \geq 4$ ;  $\text{Aut}(C)$  is  $\text{Sp}_6(2)$  and a PD-set of four elements can be found;
2.  $C^\perp$ , for  $C$  as above, is a  $[28, 7, 12]_2$ ; here  $|S| \geq 10$ ; found a PD-set of 30 elements;
3.  $C$  the  $[31, 16, 6]_5$  code with  $\text{PGL}_3(F_5)$  acting (cyclic code), the bound is 7, and a set of 14, inside a cyclic group of order 31 was found;
4. the dual of the above is a  $[31, 15, 10]_5$  code, self-orthogonal, the bound is 28, and the normalizer of a Sylow 31-subgroup has order 93. One such group was found to be a PD-set.
5.  $C$  the  $[57, 29, 8]_7$  code with  $\text{PGL}_3(F_7)$  acting (cyclic code), the bound is 15, and a set of 43 was found inside the normalizer (of order 171) of a regular cyclic group of order 57, and one of size 54 was found inside a regular cyclic group of order 57.

## Single error

Correcting a single error is, in fact, simply done by using syndrome decoding, since in that case multiples of the columns of the check matrix will give the possible syndromes. Thus the syndrome of the received vector need only be compared with the columns of the check matrix, by looking for a multiple.

## PD-sets for cyclic codes

MacWilliams [9] developed a theory for finding PD-sets for cyclic codes.

An  $[n, k, d]_q$   $C$  is cyclic if whenever  $c = c_1c_2 \dots c_n \in C$  then every cyclic shift of  $c$  is in  $C$ . Thus the mapping  $T \in S_n$  defined by

$$T : i \mapsto i + 1$$

for  $i \in \{1, 2, \dots, n\}$ , is in the automorphism group of  $C$ , and  $T^n = 1$ . If a message  $c$  is sent and  $t$  errors occur, then if  $e$  is the error vector and if there is a sequence of  $k$  zeros between two of the error positions, then  $T^j$  for some  $j$  will move the sequence of zeros into the information positions, and thus all the errors will occur in the check positions.

Thus the elements of  $\langle T \rangle$  will be a PD-set for  $C$  if  $k < \frac{n}{t}$ .



If also  $(n, q) = 1$  (i.e. the greatest common divisor of  $q$  and  $n$  is 1) then the map

$$U : i \mapsto qi$$

is also an automorphism. Such a map can improve on the largest gap between errors, i.e. on the longest sequence of zeros between errors.

Thus the group  $H = \langle T, U \rangle$  may contain a PD-set for the code.

Previous work on finding PD-sets for special types of codes includes Wolfmann [12] for Golay codes and Chabanne [4] for abelian codes. For the latter, the method uses Gröbner bases.

We try codes and designs.

# Some examples of PD-sets for codes from designs and graphs

## 1. Triangular graphs

For any  $n$ , the triangular graph  $T(n)$  is the line graph of the complete graph  $K_n$ , i.e. the vertices are the 2-subsets  $\mathcal{P}$  of  $\Omega = \{1, 2, \dots, n\}$  and vertices  $\{a, b\}$  and  $\{c, d\}$  in  $\mathcal{P}$  are adjacent if they have one letter from  $\Omega$  in common. Thus the valency is  $2(n-1)$ .

If  $A$  is an adjacency matrix for  $T(n)$  the row span of  $A$  over the field  $GF(2)$  of order 2 forms a binary code with parameters

$$\left[ \frac{n(n-1)}{2}, n-1, n-1 \right]_2$$

for  $n$  odd and

$$\left[ \frac{n(n-1)}{2}, n-2, 2(n-1) \right]_2$$

for  $n$  even, taking  $n \geq 5$  to avoid trivial cases.

## Information positions

Order the points as follows:

$$P_1 = \{1, n\}, P_2 = \{2, n\}, \dots, P_{n-1} = \{n-1, n\},$$

first, followed by the set

$$P_n = \{1, 2\}, P_{n+1} = \{1, 3\}, \dots,$$

$$P_{2n-2} = \{2, 3\}, \dots, P_{\binom{n}{2}} = \{n-2, n-1\}.$$

It can be shown that a generator matrix for  $C$  in standard form can be found with the first  $n-1$  coordinates the information symbols for  $n$  odd, and the first  $n-2$  for  $n$  even.

## PD-sets for the codes

**Result 3 (Key, Moori & Rodrigues [7])** Using for information symbols the points described above, the following sets of permutations in  $S_n$  in the natural action on the points  $\mathcal{P}$ , are PD-sets for the binary code  $C$  of the triangular graph  $T(n)$ .

1. For  $n \geq 5$  odd,

$$\mathcal{S} = \{1_G\} \cup \{(i, n) \mid 1 \leq i \leq n - 1\}$$

is a PD-set of  $n$  elements.

2. For  $n \geq 6$  and even,

$$\mathcal{S} = \{1_G\} \cup \{(i, n) \mid 1 \leq i \leq n - 1\} \cup$$

$$\{[(i, n - 1)(j, n)]^{\pm 1} \mid 1 \leq i, j \leq n - 2\}$$

is a PD-set of  $n^2 - 2n + 2$  elements.

(Recall that  $C(n)$  corrects  $\frac{n-3}{2}$  errors if  $n$  is odd, and  $n - 3$  errors if  $n$  is even.)

## Lower Bound

If  $C$  is the binary code of the triangular graph  $T(n)$  then:

- For  $n \geq 5$  odd, the lower bound for the size of a PD-set for  $C$  is  $\frac{n-1}{2}$ ;
- For  $n \geq 18$  and even, the following formula appears to hold: writing  $n = 2k + 18$ ,  $k = \frac{n-18}{2} \pmod{6} \in \{0, 1, 2, 3, 4, 5\}$ ,  $k \geq 0$ , the bound for  $n$  is

$$n - 2 + 10 \left\lfloor \frac{n - 6}{12} \right\rfloor + k + \left\lfloor \frac{k}{2} \right\rfloor.$$

The computational complexity of the decoding by this method may be quite low, of the order  $n^{1.5}$  if the elements of the PD-set are appropriately ordered. The codes are low density parity check (LDPC) codes.

## 2. Lattice graphs $L_2(n)$

The **line graph** of a graph  $\Gamma = (V, E)$  is the graph  $\Gamma^t = (E, V)$  where  $e$  and  $f$  are adjacent in  $\Gamma^t$  if  $e$  and  $f$  share a vertex in  $\Gamma$ . The **complete bipartite graph**  $K_{n,n}$  on  $2n$  vertices with  $n^2$  edges, has for line graph, the **lattice graph**  $L_2(n)$ , which has vertex set the set of ordered pairs  $\{(i, j) \mid 1 \leq i, j \leq n\}$ , where two pairs are adjacent if and only if they have a common coordinate.  $L_2(n)$  is strongly regular of type  $(n^2, 2(n - 1), n - 2, 2)$ .

Let  $n \geq 2$  be any integer and let  $L_2(n)$  denote the lattice graph with vertex set  $\mathcal{P}$  the  $n^2$  ordered pairs  $(i, j)$ ,  $1 \leq i, j \leq n$ . The 1-design  $\mathcal{D} = (\mathcal{P}, \mathcal{B})$  will have point set  $\mathcal{P}$  and for each point  $(i, j) \in \mathcal{P}$ ,  $1 \leq i, j \leq n$ , a block, which we denote by  $\overline{(i, j)}$ , is defined in the following way:

$$\overline{(i, j)} = \{(i, k) \mid k \neq j\} \cup \{(k, j) \mid k \neq i\}.$$

Then the block set is

$$\mathcal{B} = \{ \overline{(i, j)} \mid 1 \leq i, j \leq n \}.$$

The incidence vector of the block  $\overline{(i, j)}$  is

$$v^{\overline{(i, j)}} = \sum_{k \neq j} v^{(i, k)} + \sum_{k \neq i} v^{(k, j)} = \sum_{k=1}^n v^{(i, k)} + \sum_{k=1}^n v^{(k, j)} \quad (1)$$

where the incidence vector of the subset  $X \subseteq \mathcal{P}$  is denoted by  $v^X$ , but writing  $v^{(i, j)}$  instead of  $v^{\{(i, j)\}}$ . To avoid trivial cases we will take  $n \geq 5$ .



**Result 4** For  $n \geq 5$ , the automorphism group of the lattice graph  $L_2(n)$  is  $S_n \wr S_2$ , the wreath product of  $S_n$  with  $S_2$ . The binary code formed by the row space over  $F_2$  of an adjacency matrix for  $L_2(n)$  is a  $[n^2, 2(n-1), 2(n-1)]_2$  code with  $S_n \wr S_2$  acting as an automorphism group.

**Lemma 1** For every  $\sigma \in S_n$ , the vector

$$w(\sigma) = \sum_{i=1}^n v^{(i, i\sigma)} \quad (2)$$

is in  $C^\perp$ .

**Proof:** Let  $v^{\overline{(i,j)}} \in C$  and  $\sigma \in S_n$ . Then

$$\begin{aligned} (w(\sigma), v^{\overline{(i,j)}}) &= \left( \sum_{m=1}^n v^{(m, m\sigma)}, \sum_{k \neq j} v^{(i, k)} + \sum_{k \neq i} v^{(k, j)} \right) \\ &= (v^{(i, i\sigma)}, \sum_{k \neq j} v^{(i, k)}) + (v^{(j\sigma^{-1}, j)}, \sum_{k \neq i} v^{(k, j)}). \end{aligned}$$

Now  $(v^{(i, i\sigma)}, \sum_{k \neq j} v^{(i, k)}) = 0$  if  $i\sigma = j$ , i.e. if  $j\sigma^{-1} = i$ , which implies that  $(v^{(j\sigma^{-1}, j)}, \sum_{k \neq i} v^{(k, j)}) = 0$ . Conversely,  $(v^{(i, i\sigma)}, \sum_{k \neq j} v^{(i, k)}) = 1$  if  $i\sigma \neq j$ , i.e. if  $j\sigma^{-1} \neq i$ , which implies that  $(v^{(j\sigma^{-1}, j)}, \sum_{k \neq i} v^{(k, j)}) = 1$ . Thus  $(w(\sigma), v^{\overline{(i,j)}}) = 0$ , as required. ■

**Lemma 2** *The minimum weight of  $C^\perp$  for  $n \geq 5$  is 4.*

**Proof:** For any  $\sigma \in S_n$ ,  $w(\sigma) \in C^\perp$  by Lemma 1. For any  $i, j \in \Omega$ , where  $i \neq j$ , write

$$w(i, j; \sigma) = w(\sigma) + w((i, j)\sigma) = v^{(i, i\sigma)} + v^{(i, j\sigma)} + v^{(j, i\sigma)} + v^{(j, j\sigma)}, \quad (3)$$

so that  $C^\perp$  has words of weight 4.

Consideration of the various possibilities shows easily that  $C^\perp$  cannot have words of weight less than 4. ■

In the special case where  $\sigma = (i, k)(j, l)$ , where  $k, l \in \Omega$ ,  $k \neq l$ , using the notation of (3), we write

$$u(\{i, j\}; \{k, l\}) = w(i, j; (i, k)(j, l)) = v^{(i,k)} + v^{(i,l)} + v^{(j,k)} + v^{(j,l)}, \quad (4)$$

since the ordering of  $i$  and  $j$  or of  $k$  and  $l$  is arbitrary.

**Lemma 3** *A sequence  $\mathcal{U}$  of weight-4 vectors  $u(i, j; k, l)$  can be found such that  $\mathcal{U}$  together with  $j$  forms a basis for  $C^\perp$  when  $n$  is odd, and  $\mathcal{U}$  together with  $w(\sigma)$  where  $\sigma = (1, 2, \dots, n)$  forms a basis when  $n$  is even.*

The point ordering is as follows: first we take, in order,

$$(1, 1), (1, 2), \dots, (1, n-1), (2, 1), (2, 2), \dots, (n-1, 1), \dots, (n-1, n-1), \quad (5)$$

giving  $(n-1)^2$  points, followed by

$$(1, n), (2, n), \dots, (n-1, n), (n, 1), (n, 2), \dots, (n, n), \quad (6)$$

for the remaining  $2n-1$  points. Since the dimension of  $C^\perp$  is  $(n-1)^2 + 1$ , we need a further element and we will show that the first point,  $(1, n)$ , of Equation (6) can be included in the information set for  $C^\perp$ .

## PD-sets for the codes

**Result 5 (Key & Seneviratne [8])** For  $n \geq 5$ , let  $C$  be the  $[n^2, 2(n-1), 2(n-1)]_2$  binary code from the row span of an adjacency matrix for the lattice graph  $L_2(n)$ . Then a PD-set of  $n^2$  elements can be found for  $C$ . Using the  $2(n-1)$  points (ordered pairs)

$$\{(i, n) \mid 2 \leq i \leq n-1\} \cup \{(n, i) \mid 1 \leq i \leq n\}$$

as information symbols, the set

$$\mathcal{S} = \{((i, n), (j, n)) \mid 1 \leq i \leq n, 1 \leq j \leq n\}$$

of permutations in  $S_n \times S_n$ , in the natural action on the points (ordered pairs), forms a PD-set of size  $n^2$  for  $C$ .

Here  $(n, n)$  denotes the identity element of  $S_n$ .

Recall that the code corrects  $t = n - 2$  errors.

## Proof of Result 5

Denote the information symbols by  $\mathcal{I}$ . Now  $C$  can correct  $t = n - 2$  errors, so we need to show that every set of  $s \leq t$  points can be moved by some element of  $\mathcal{S}$  into the check positions  $\mathcal{E}$ . Let

$$\mathcal{T} = \{(a_1, b_1), (a_2, b_2), \dots, (a_s, b_s)\}$$

be a set of  $s \leq t = n - 2$  points of  $\mathcal{P}$ .

Let  $\Omega_1 = \{a_1, a_2, \dots, a_s\}$  and  $\Omega_2 = \{b_1, b_2, \dots, b_s\}$ .

Then  $|\Omega_i| \leq n - 2$  for  $i = 1, 2$  and thus we can find  $k \neq n$  and  $l \neq n$  such that  $k \notin \Omega_1$  and  $l \notin \Omega_2$ . Then

$$g = ((k, n), (l, n)) \in \mathcal{S}_n \times \mathcal{S}_n$$

will satisfy  $\mathcal{T}^g \subseteq \mathcal{E}$ . Thus  $\mathcal{S}$  forms a PD-set for the code. ■

## Lower bound

Using Magma [3] we found that the following formulae for the lower bound (Result 2) appear to hold, showing that the bound is linear in  $n$ :

■ for  $n$  odd:

$$n \equiv 9 \pmod{12}, n \geq 21 : \frac{1}{2}(5n - 11) - 2\lfloor \frac{n-3}{4} \rfloor - \lceil \frac{n-5}{6} \rceil = \frac{1}{6}(11n - 15);$$

$$n \not\equiv 9 \pmod{12}, n \geq 29 : \frac{1}{2}(5n - 11) - 2\lfloor \frac{n-3}{4} \rfloor - \lfloor \frac{n-5}{6} \rfloor;$$

■ for  $n \geq 12$  even:

$$2n - 3 - \lceil \frac{n-2}{4} \rceil + \lfloor \frac{n-6}{12} \rfloor.$$

### 3. Lattice graphs $L_2(m, n)$ , $m < n$

The lattice graph  $L_2(m, n)$  for  $m < n$  has vertex set the set  $\mathcal{P}$  of ordered pairs  $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ , where two pairs are adjacent if and only if they have a common coordinate. It is a regular graph of valency  $m + n - 2$ .

**Result 6** For  $5 \leq m \leq n$  the automorphism group of the lattice graph  $L_2(m, n)$  contains  $S_m \times S_n$ . If  $C$  is the binary code formed by the row space over  $F_2$  of an adjacency matrix for  $L_2(m, n)$  for  $m < n$ , then  $C$  is

- $[mn, m + n - 2, 2m]_2$  for  $m + n$  even;
- $[mn, m + n - 1, m]_2$  for  $m + n$  odd.



## PD-sets for the codes

**Result 7** Let  $C$  be the binary code of the lattice graph  $L_2(m, n)$  for  $5 \leq m < n$ ,

$$\mathcal{I} = \{(i, n) \mid 1 \leq i \leq m\} \cup \{(m, i) \mid 1 \leq i \leq n - 1\},$$

and

$$\mathcal{S} = \{((i, m), (j, n)) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then  $\mathcal{S}$  is a PD-set of  $mn$  elements for  $C$  using  $\mathcal{I}$  as information symbols for  $m + n$  odd, and  $\mathcal{I} \setminus \{(1, n)\}$  for  $m + n$  even.

The existence of a PD-set for a code is not invariant under equivalence: for example, in Result ??, the points

$$\{1, 2, n\}, \{1, 3, n\}, \dots, \{n - 2, n - 1, n\}$$

can be taken as information symbols, but if they are, the code will not have a PD-set to correct all the allowed errors.

**Example:** For  $n = 9$  the  $[84, 14, 7]_2$  code corrects  $t = 3$  errors but there is no element in  $S_9$  that move the three points

$$\{1, 2, 9\}, \{3, 4, 5\}, \dots, \{6, 7, 8\}$$

into the check positions. However, if  $\{7, 8, 9\}$  is placed in the check positions and  $\{6, 7, 8\}$  in the information positions, then it can be done: in this example,  $(6, 9)$  will do it.

## 4. Projective planes

The desarguesian finite projective planes,  $\Pi = PG_2(F_q)$ , where  $q = p^t$ , give codes over  $F_p$  with dimension  $\binom{p+1}{2}^t + 1$  and minimum weight  $q + 1$ . The codes are subfield subcodes of the projective generalized Reed-Muller codes, and the automorphism groups are  $PGL_3(q)$ . In addition the codes are all cyclic, and the groups doubly transitive.

Thus PD-sets that correct two errors always exist. To find them, or to find PD-sets that will correct all the errors the code is capable of correcting, we need suitable information positions for these codes. Earlier work on finding bases for the codes (by Moorhouse [10] and Blokhuis and Moorhouse [2]) gives bases when  $q = p$ . Work in progress by Jirapha Limbupasiriporn has a tentative basis for  $q = p^2$ . With these bases, using for information positions the corresponding points in homogeneous coordinates, we can produce PD-sets to correct two errors in all cases.

## Partial PD-set for projective planes of prime order

Let  $\Pi = PG_2(F_p)$  where  $p$  is a prime, and let  $C = C_p(\Pi)$ . Then

$$C = [p^2 + p + 1, \binom{p+1}{2} + 1, p + 1]_p.$$

For  $0 \leq s \leq p - 1$  let

$$\mathcal{C}_s = \{(0, 1, s)\} \cup \{(1, k, ks) \mid 1 \leq k \leq p - (s + 1)\},$$

and

$$\mathcal{I} = \bigcup_{s=0}^{p-1} \mathcal{C}_s \cup \{(1, 0, 0)\}.$$

Then  $\mathcal{I}$  can be taken as the information symbols for  $C$  (by Moorhouse [10]) and the following set of matrices  $S$  will form a PD-set that will correct two errors.

Define, for  $r, g, i \in F_p = F$ ,

$$B_{r,g,i} = \begin{bmatrix} 0 & 0 & 1 \\ 1 - rg & -r & 1 - ri \\ g & 1 & i \end{bmatrix}$$

and

$$C_{r,g,i} = \begin{bmatrix} 1 & 0 & 0 \\ -rg & -r & 1 - ri \\ g & 1 & i \end{bmatrix}.$$

Let

$$S = \{C_{r,0,0}, C_{r,\pm 1,i}, C_{r,0,-1}, C_{r,1,-1} \mid r \in F, i \neq -1 \in F\} \cup \\ \{B_{r,0,0}, B_{r-1,0,-1}, B_{0,-1,1}, B_{0,-1,2} \mid r \in F\} \cup \{I_3\}.$$

$S$  will act as a PD-set to correct two errors, and  $S$  contains  $p^2 + 3p + 3$  elements.

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