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Special LCD codes from products of graphs

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Abstract We examine the binary codes from the adjacency matrices of various products of graphs, and show that if the binary codes of a set of graphs have the property that their dual codes are the codes of the associated reflexive graphs, and are thus LCD, i.e. have zero hull, then, with some restrictions, the binary code of the product will have the same property. The codes are candidates for decoding using this property, or also, in the case of the direct product, by permutation decoding.

Keywords LCD codes · codes from graphs · products of graphs

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1 Introduction

Various products of graphs are defined and discussed in [9]. We will examine some of these products of graphs for the property of their binary codes being $RLCD$ (see [16]) if the binary codes of their component graphs are $RLCD$. Here a code C from the row span of an adjacency matrix for a graph is said to be $RLCD$ if the code from the row span of the corresponding reflexive

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graph (i.e. including all loops) is the dual code, $C^{\perp}$, so this implies that C is LCD. A code from an adjacency matrix of a graph that is $RLCD$ is useful for decoding purposes, not only from the method for LCD codes as described by Massey [24], but also from a method described specifically for $RLCD$ codes in [17].

All the graphs will be undirected. In addition, in considering any of these products of n undirected graphs $\Gamma_i = (V_i, E_i)$, the vertex set of the product will be the cartesian product of the sets of vertices V_i, i.e. $V_1 \times V_2 \times \ldots \times V_n$. Adjacency, and hence edge sets, are defined differently for the various products.

A summary of our results addressing this problem for the most common of these products is the following theorem which is proved as Propositions 1, 2 and 3 in the following sections:

Theorem 1 *Let Γ be the graph product of the n graphs Γ_i, for $i = 1, \ldots, n$, where the product is the Cartesian product, $\Box$, the Direct (Categorical) product, $\times$, or the Strong product, $\boxtimes$, of the graphs.*

If all the the binary codes $C_2(\Gamma_i)$ are $RLCD$, then so is $C_2(\Gamma)$.

Some recent papers involving codes associated with graphs, and in particular, LCD codes, although not necessarily $RLCD$, can be found, for example, in the following: [6,18,19,27].

The full definition of $RLCD$ is given in Definition 2 in Section 2, where some other related concepts are defined, as well as some background results. Theorem 1 holds with some modifications for the other graph products examined. In addition, in the case of the direct product of graphs in Section 4, it is possible to obtain s-PD-sets for the code of the product if such sets are known for the codes of the individual graphs: see Lemma 8 and Proposition 8 for the triangular graphs. In the case of the direct product more can be said about the parameters of the binary code of the product, and these results are summarized in Theorem 2.

The definitions of the various graph products are given in Sections 3, 4, 5, 6 and 7, and in each of these cases, other properties of the binary code of the product are examined, including minimum weight, information sets, and the possibility of using permutation decoding. Section 8 has some examples using graphs whose binary codes are known to be $RLCD$, in particular the triangular graphs and the Paley graphs.

2 Background

2.1 Definitions and previous results

Basic definitions not covered here can be found in [1], or see also [28,29] for other concepts related to designs, codes and graphs.

The **graphs**, $\Gamma = (V, E)$ with vertex set V and edge set E, discussed in this work are undirected with no loops, apart from the case where **all** loops are included, in which case the graph is called the **reflexive** associate of Γ, denoted by $R\Gamma$. If $x, y \in V$ and x and y are adjacent, we write $x \sim y$, and xy for the **edge** in E that they define. The **complementary** graph is denoted by $\overline{\Gamma} = (V, \overline{E})$ where for $x, y \in V$, $x \neq y$, $x \sim y$ in Γ if and only if $x \not\sim y$ in $\overline{\Gamma}$. The **set of neighbours** of $x \in V$ is denoted by $N(x)$, and the **valency of** x is $|N(x)|$. Γ is **regular** if all the vertices have the same valency.

An **adjacency** matrix $A = [a_{x,y}]$ for Γ is a symmetric $|V| \times |V|$ matrix with rows and columns labelled in the same order by the vertices $x, y \in V$, and with $a_{x,y} = 1$ if $x \sim y$ in Γ, and $a_{x,y} = 0$ otherwise. Then $RA = A + I$ is an adjacency matrix for $R\Gamma$, and $\overline{A} = J - I - A$ one for $\overline{\Gamma}$, where $I = I_{|V|}$ and J is the $|V| \times |V|$ all-ones matrix. The row corresponding to $x \in V$ in A will be denoted by r_x, that in RA by s_x, and that in $\overline{A}$ by c_x.

The codes here are **linear codes**, and the notation $[n, k, d]_q$ will be used for a q-ary code C of length n, dimension k, and minimum weight d, where the **weight** $\mathbf{wt}(v)$ of a vector v is the number of non-zero coordinate entries. The **code** over a field F of a graph $\Gamma = (V, E)$ is the row span over F of an adjacency matrix A for Γ, and written as $C_F(A)$, $C_F(\Gamma)$, or $C_p(A)$, $C_p(\Gamma)$, respectively, if $F = \mathbb{F}_p$. If $S \subseteq V$, the **incidence vector** of S is denoted by v^S.

Notation 1 By abuse of language, we will also use r_x (respectively s_x) to denote the set of neighbours of x, $N(x) = \{y \in V \mid x \sim y\}$ (respectively $N(x) \cup \{x\}$). Furthermore, we shall be dealing with different graphs in this paper and use the same notation r_x (respectively s_x) for any of the graphs, with the understanding that $x \in V$ for the particular graph under consideration, so that the notation will be unambiguous. We will also use r_x (respectively s_x) to denote the word in the code, i.e. as a row of the matrix. This should also be clear.

The **uniform subset graph** $\Gamma(n, k, r)$ has for vertices $V = \Omega^{\{k\}}$, the set of all subsets of size k of a set of size n, with two k-subsets x and y defined to be adjacent if $|x \cap y| = r$. The valency of $\Gamma(n, k, r)$ is $\binom{k}{r}\binom{n-k}{k-r}$.

A graph $\Gamma = (V, E)$, neither complete nor null, is **strongly regular** of type (n, k, λ, μ) if it is regular on $n = |V|$ vertices, has valency k, and is such that any two adjacent vertices are together adjacent to λ vertices and any two non-adjacent vertices are together adjacent to μ vertices.

2.2 *LCD* codes

Definition 1 A linear code C over any field is an LCD code **(linear code with complementary dual)** if $\mathrm{Hull}(C) = C \cap C^{\perp} = \{0\}$.

If C is an LCD code of length n over a field F, then $F^n = C \oplus C^{\perp}$. Thus the **orthogonal projector map** Π_C from F^n to C can be defined as follows: for $v \in F^n$,

$$v\Pi_C = \begin{cases} v \text{ if } v \in C, \\ 0 \text{ if } v \in C^{\perp} \end{cases}, \tag{1}$$

and Π_C is defined to be linear. [1] This map is only defined if C (and hence also $C^\perp$) is an LCD code. Similarly then $\Pi_{C^\perp}$ is defined.

Note that for all $v \in F^n$,

$$v = v\Pi_C + v\Pi_{C^\perp}. \tag{2}$$

We will use [24, Proposition 4]:

Result 1 (Massey) *Let C be an LCD code of length n over the field F and let φ be a map $\varphi : C^\perp \mapsto C$ such that $u \in C^\perp$ maps to one of the closest codewords v to it in C. Then the map $\tilde{\varphi} : F^n \mapsto C$ such that*

$$\tilde{\varphi}(r) = r\Pi_C + \varphi(r\Pi_{C^\perp})$$

maps each $r \in F^n$ to one of it closest neighbours in C. [2]

We make the following observation which will be of use in the next section:

Lemma 1 *If C is a q-ary code of length n such that $C + C^\perp = \mathbb{F}_q^n$ then C is LCD.*

Proof: Since $(C + C^\perp)^\perp = C^\perp \cap C = (\mathbb{F}_q^n)^\perp = \{0\} = \mathrm{Hull}(C)$, C (and $C^\perp$) are LCD. ■

Note then that if $C = C_p(\Gamma)$ and $RC = C_p(R\Gamma)$ for a graph Γ on n vertices, p a prime, then $C + RC = \mathbb{F}_p^n$, so if $RC = C^\perp$, then C is LCD.

From [16]:

Definition 2 Let $\Gamma = (V, E)$ be a graph with adjacency matrix A. Let p be any prime, $C = C_p(A)$, $RC = C_p(RA)$ (for the reflexive graph), and $\overline{C} = C_p(\overline{A})$. Then

- if $C = RC^\perp$, then we call C a **reflexive** LCD code, and write $RLCD$ for such a code;
- if Γ is regular and $C = \overline{C}^\perp$, then we call C a **complementary** LCD code, and write $CLCD$ for such a code.

We note the following result from [8], which is given there for $p = 2$ but it holds for all primes p, so we state it for all p:

Result 2 (Proposition 2.2 [8]) *If A is a symmetric integral matrix, and C_A, C_{A+I} denote the row span over $\mathbb{F}_p$, where p is a prime, of A, $A+I$ respectively, then $C_A^\perp \subseteq C_{A+I}$ with equality if and only if $A(A + I) \equiv 0 \pmod{p}$.*

The following two results are lemmas in [16]:

Result 3 *If $\Gamma = (V, E)$ is regular of valency ν, $|V| = n$, p is a prime, then both $C_p(\Gamma)$ and $C_p(\overline{\Gamma})$ can be $RLCD$ if and only if $(n - 2\nu - 1) \equiv 0 \pmod{p}$.*

[1] Note typographical error on p.338, l.-11, in [24]

[2] Note typographical error on p.341, l.-7, in [24]

Note 1 If we know the eigenvalues of A, and if they are integral, we can use them to get information regarding the possible dimension of the codes C and RC. Since if λ is an eigenvalue for a matrix M then $\lambda + 1$ is an eigenvalue for $M + I$, this will also give information about RC. If M is a $v \times v$ integral matrix with integral eigenvalues, then modulo p these will still be eigenvalues, but not necessarily all distinct. If none or at most one reduce to 0 modulo p then the p-rank of M will be v or $v - m_j$, respectively, where m_j is the multiplicity of the eigenvalue that is zero. In any case, the dimension of the zero eigenspace over $\mathbb{F}_p$ of the matrix A or $A + I$ is at most the sum m of the multiplicities of the eigenvalues that reduce to 0 modulo p, and thus the p-rank of A or $A + I$ is at least $v - m$.

From [16, Lemma 3]:

Result 4 *Let $\Gamma = (V, E)$ be a graph with adjacency matrix A that has integral eigenvalues and suppose p is a prime for which $C_p(\Gamma)$ is RLCD. Then* $\dim(C_p(\Gamma))$ *is the sum of the multiplicities of the eigenvalues that are non-zero modulo p.*

A special decoding method for *RLCD* binary codes is given in [17, Lemmas 1,2], and the discussion in that paper following those lemmas.

A summary of the algorithm for such decoding is as follows, and it assumes that the system allows at most s errors where $s \leq t$, the maximum number of errors nearest-neighbour decoding allows: suppose $C = C_2(\Gamma)$, where $\Gamma = (V, E)$, is *RLCD* and has minimum distance d and $t = \lfloor \frac{d-1}{2} \rfloor$, and the transmitted word from C has no more than t errors. Let $|V| = n$. Then

- Compute separately all the sums $\sum_{x \in K} s_x$ for every subset $K \subset V$ of size k where $1 \leq k \leq t$. Let $\mathcal{S}_k = \{\sum_{x \in K} s_x \mid K \subset V, |K| = k\}$, for $1 \leq k \leq t$.
- Suppose $w = v^S$ is the received word and that $s \leq t$ errors have occurred. Form the sum $v = \sum_{x \in S} s_x$.
- If $v = 0$ then no errors have occurred. If $v \neq 0$ then check the sets $\mathcal{S}_k$ to see if $v \in \mathcal{S}_k$, starting with $k = 1$ and then increasing k to s or at most t.
- When a set J is found such that $v = \sum_{x \in J} s_x$, decode as $\sum_{x \in S} r_x + \sum_{x \in J} r_x = v^S + v^J$.

The worst case complexity for t errors is $\mathcal{O}(n^{t+1})$. For a small number of errors s this could be feasible.

2.3 Permutation decoding

Permutation decoding was first developed by MacWilliams [22] and involves finding a set of automorphisms of a code called a PD-set. The method is described fully in MacWilliams and Sloane [23, Chapter 16, p. 513] and Huffman [10, Section 8]. In [12] and [21] the definition of PD-sets was extended to that of s-PD-sets for s-error-correction:

Definition 3 If C is a t-error-correcting code with information set $\mathcal{I}$ and check set $\mathcal{C}$, then a **PD-set** for C is a set $\mathcal{S}$ of automorphisms of C which is such that every t-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into the check positions $\mathcal{C}$.

For $s \leq t$ an s**-PD-set** is a set $\mathcal{S}$ of automorphisms of C which is such that every s-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into $\mathcal{C}$.

The algorithm for permutation decoding is as follows: we have a t-error-correcting $[n,k,d]_q$ code C with check matrix H in standard form. Thus the generator matrix $G = [I_k|A]$ and $H = [-A^T|I_{n-k}]$, for some A, and the first k coordinate positions correspond to the information symbols. Any vector v of length k is encoded as vG. Suppose x is sent and y is received and at most t errors occur. Let $S = \{g_1, \dots, g_s\}$ be the PD-set. Compute the syndromes $H(yg_i)^T$ for $i = 1, \dots, s$ until an i is found such that the weight of this vector is t or less. Compute the codeword c that has the same information symbols as yg_i and decode y as cg_i^{-1}.

Notice that this algorithm actually uses the PD-set as a sequence. Thus it is expedient to index the elements of the set S by the set $\{1, 2, \dots, |S|\}$ so that elements that will correct a small number of errors occur first. Thus if **nested** s**-PD-sets** are found for all $1 < s \leq t$ then we can order S as follows: find an s-PD-set S_s for each $0 \leq s \leq t$ such that $S_0 \subset S_1 \ldots \subset S_t$ and arrange the PD-set S as a sequence in this order:

$$S = [S_0, (S_1 - S_0), (S_2 - S_1), \dots, (S_t - S_{t-1})].$$

(Usually one takes $S_0 = \{id\}$.)

There is a bound on the minimum size that a PD-set S may have, due to Gordon [7], from a formula due to Schönheim [26], and quoted and proved in [10]:

Result 5 *If S is a PD-set for a t-error-correcting $[n,k,d]_q$ code C, and $r = n - k$, then*

$$|S| \geq \left\lceil \frac{n}{r} \left\lceil \frac{n-1}{r-1} \left\lceil \dots \left\lceil \frac{n-t+1}{r-t+1} \right\rceil \dots \right\rceil \right\rceil \right\rceil = G(t). \qquad (3)$$

This result can be adapted to s-PD-sets for $s \leq t$ by replacing t by s in the formula and $G(s)$ for $G(t)$.

We note the following result from [14, Lemma 1]:

Result 6 *If C is a t-error-correcting $[n,k,d]_q$ code, $1 \leq s \leq t$, and S is an s-PD-set of size $G(s)$ then $G(s) \geq s + 1$. If $G(s) = s + 1$ then $s \leq \lfloor \frac{n}{k} \rfloor - 1$.*

In [13, Lemma 7] the following was proved:

Result 7 *Let C be a linear code with minimum weight d, $\mathcal{I}$ an information set, $\mathcal{C}$ the corresponding check set and $\mathcal{P} = \mathcal{I} \cup \mathcal{C}$. Let G be an automorphism group of C, and n the maximum value of $|\mathcal{O} \cap \mathcal{I}|/|\mathcal{O}|$, over the G-orbits $\mathcal{O}$. If $s = \min(\lceil \frac{1}{n} \rceil - 1, \lfloor \frac{d-1}{2} \rfloor)$, then G is an s-PD-set for C.*

This result holds for any information set. If the group G is transitive then $|\mathcal{O}|$ is the degree of the group and $|\mathcal{O} \cap \mathcal{I}|$ is the dimension of the code.

The worst-case time complexity for the decoding algorithm using an s-PD-set of size z on a code of length n and dimension k is $\mathcal{O}(nkz)$.

2.4 Kronecker product of matrices

The adjacency matrices of products of graphs are conveniently described in terms of Kronecker products of matrices, so we give a brief background of this product.

The Kronecker product is a special case of the tensor product.

The Kronecker product of two matrices A and B is denoted by $A \otimes B$ and if A is $m \times n$ and B is $p \times q$ then $A \otimes B$ is $mp \times nq$. If $A = [a_{ij}]$ then

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Properties of Kronecker products

Assuming products where written are defined and that k is a scalar:

$A \otimes (B + C) = A \otimes B + A \otimes C$; $\quad (A + B) \otimes C = A \otimes C + B \otimes C$;
$kA \otimes B = A \otimes (kB) = k(A \otimes B)$; $\quad (A \otimes B) \otimes C = A \otimes (B \otimes C)$;
$(A \otimes B)(C \otimes D) = AC \otimes BD$; $\quad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
$\text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B)$; $\quad I_n \otimes I_m = I_{nm}$.

Furthermore, if A is $n \times n$ and B is $m \times m$, and λ_i for $i = 1, \ldots, n$ the eigenvalues of A, and μ_i for $i = 1, \ldots, m$ those of B, then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ for $i = 1, \ldots, n, j = 1, \ldots, m$.

Note 2 : In the following, graphs are defined from graphs $\Gamma_i = (V_i, E_i)$, $i = 1, \ldots, n$, to have vertex set $V_1 \times \ldots \times V_n$. If $\alpha_i \in \text{Aut}(\Gamma_i)$, $i = 1, \ldots, n$, then $(\alpha_1, \ldots, \alpha_n)$ defined by

$$(\alpha_1, \ldots, \alpha_n) :< x_1, \ldots, x_n > \mapsto < x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} > \tag{4}$$

is an automorphism of the graph defined on the vertex set $V_1 \times \ldots \times V_n$.

Since we will be using adjacency matrices, we will need an ordering on the vertices of the vertex set V_i of each each of the graphs $\Gamma_i = (V_i, E_i)$. For the vertex set of the graph product, $V_1 \times \ldots \times V_n$, we use lexicographical ordering, i.e. dictionary reading from left to right. Thus, for example for $n = 2$, $|V_1| = m$, $|V_2| = k$, $V_1 = \{x_1, \ldots, x_m\}$ and $V_2 = \{y_1, \ldots, y_k\}$, as ordered sets, then the ordering for $V_1 \times V_2$ is

$$\{< x_1, y_1 >, < x_1, y_2 >, \ldots < x_1, y_k >, < x_2, y_1 >, \ldots, < x_2, y_k >, \ldots,$$
$$< x_m, y_1 >, \ldots, < x_m, y_k >\}.$$

3 Cartesian products of graphs $\Gamma_1 \Box \Gamma_2$

If $\Gamma_i = (V_i, E_i)$ for $i = 1, 2$ are graphs with $|V_i| = n_i$ and adjacency matrix A_i then $\Gamma_1 \Box \Gamma_2$ will denote the cartesian product of the graphs, with vertex set $V = V_1 \times V_2$. Here if $< x, y >, < u, v > \in V$, then

- adjacency is defined by $< x, y >\sim< u, v >$ in Γ if and only if $x = u$ and $y \sim v$ in Γ_2, or $y = v$ and $x \sim u$ in Γ_1;
- if Γ_1 and Γ_2 are regular of valency ν_1, ν_2 respectively, then $\Gamma_1 \Box \Gamma_2$ is regular of valency $\nu_1 + \nu_2$;
- an adjacency matrix for $\Gamma_1 \Box \Gamma_2$ is given by

$$A_{1\Box 2} = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2.$$

Note 3 From [16,6] and Result 2 we know that a code $C_p(\Gamma)$ is $RLCD$ if and only if an adjacency matrix A for Γ satisfies $A^2 = -A$ over $\mathbb{F}_p$. Clearly this implies that the null graph, i.e. the complement of the complete graph K_n, which has the zero code over any field, is thus $RLCD$. We will exclude this graph from our discussions, i.e. we assume that the Γ_i are not null.

Proposition 1 *Let $\Gamma_i = (V_i, E_i)$ for $i = 1, \ldots, n$ be graphs with $|V_i| = n_i$ and adjacency matrix A_i. Let Γ be the cartesian product $\Box_{i=1}^n \Gamma_i$. Then $C_2(\Gamma)$ is $RLCD$ if $C_2(\Gamma_i)$ is $RLCD$ for each $i = 1, \ldots, n$. If precisely one of the $C_2(\Gamma_i)$ is not $RLCD$ and all the others are, then $C_2(\Gamma)$ is not $RLCD$.*

Proof: We need only prove this for the cartesian product of two graphs. With the same notation as above, we have, for an adjacency matrix $A_{1\Box 2}$ for $\Gamma_1 \Box \Gamma_2$,

$$A_{1\Box 2} = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2.$$

Suppose $C_2(\Gamma_i)$ for $i = 1, 2$ is $RLCD$. Then $A_i^2 = -A_i = A_i$, $i = 1, 2$, over $\mathbb{F}_2$. By the rules of multiplication of Kronecker products of matrices,

$$\begin{aligned} A_{1\Box 2}^2 =&(A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2)^2 \\ =&(A_1 \otimes I_{n_2})^2 + (I_{n_1} \otimes A_2)^2 + (A_1 \otimes I_{n_2})(I_{n_1} \otimes A_2) + (I_{n_1} \otimes A_2)(A_1 \otimes I_{n_2}) \\ =&(A_1^2 \otimes I_{n_2}) + (I_{n_1} \otimes A_2^2) + 2(A_1 \otimes A_2) \\ =& A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 = A_{1\Box 2}, \end{aligned}$$

and so $C_2(\Gamma_1 \Box \Gamma_2)$ is $RLCD$.

Now suppose $C_2(\Gamma)$ is $RLCD$ where $\Gamma = \Gamma_1 \Box \Gamma_2$. Suppose $C_2(\Gamma_1)$ is not $RLCD$, but $C_2(\Gamma_2)$ is $RLCD$. We have

$$A_{1\Box 2}^2 - A_{1\Box 2} = (A_1^2 - A_1) \otimes I_{n_2} + I_{n_1} \otimes (A_2^2 - A_2) = (A_1^2 - A_1) \otimes I_{n_2} = 0.$$

But according to the properties of Kronecker products, $\text{rank}_2(A \otimes B) = \text{rank}_2(A)\text{rank}_2(B)$, so $\text{rank}_2(A_1^2 - A_1) = 0$ and hence $A_1^2 - A_1 = 0$, and $C_2(\Gamma_1)$ is $RLCD$.

Clearly this can be extended to any number of components in the product. ■

Note 4 If all the Γ_i are equal to a graph Γ then $\Box_{i=1}^n \Gamma$ is written $\Gamma^{\Box, n}$.

Lemma 2 *Let $\Gamma = (V, E)$ be regular of valency ν, A an adjacency matrix. If $C_p(\Gamma)$ is $RLCD$ and $p|\nu$, then $C_p(\Gamma) \subseteq C_p(\overline{\Gamma})^{\perp}$. If in addition Γ has integral eigenvalues and $p \not| (|V| - 1)$, then $C_p(\Gamma)$ is $CLCD$.*

Proof: We have $A^2 = -A$, so $A\overline{A} = A(J - I - A) = AJ - (A + A^2) = AJ = \nu J = 0$. The second statement follows from [16, Proposition 2]. ■

Lemma 3 *If Γ_i, $i = 1, 2$ are regular of valency ν_i, $i = 1, 2$ respectively, and if both $C_2(\Gamma_i)$ for $i = 1, 2$ are CLCD, then if both ν_1, ν_2 are even, $C_2(\Gamma_1 \Box \Gamma_2)$ is RLCD. If in addition at least one of $|V_1|$,$|V_2|$ is even, and Γ_i for $i = 1, 2$ have integral eigenvalues, then $C_2(\Gamma_1 \Box \Gamma_2)$ is CLCD.*

Proof: If A_i, $i = 1, 2$, is an adjacency matrix for Γ_i then $A_i(J - I - A_i) = 0 = \nu_i J - A_i - A_i^2$. So if ν_i is even, $A_i = -A_i^2$, and $C_2(\Gamma_i)$ are both $RLCD$, and thus so is $C_2(\Gamma_1 \Box \Gamma_2)$. By Lemma 2, since the valency of $\Gamma_1 \Box \Gamma_2$ is $\nu_1 + \nu_2$, which is even, $C_2(\Gamma_1 \Box \Gamma_2)$ is $CLCD$ if $2 \not| (|V_1||V_2|| - 1)$.

Corollary 1 *If $\Gamma = \Box_{i=1}^n \Gamma_i$ where $C_2(\Gamma_i)$ is CLCD for $i = 1, \ldots, n$ and all the $\Gamma_i = (V_i, E_i)$ are regular of even valency, then $C_2(\Gamma)$ is RLCD, and if in addition, all the eigenvalues are integral and at least one of the $|V_i|$ is even, then $C_2(\Gamma)$ is CLCD.*

Lemma 4 *Let $\Gamma = \Gamma_1 \Box \Gamma_2$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, 2$. Let $w_i \in C_2(\Gamma_i)^\perp$ be of weight d_i, with $S_1 = \mathrm{Supp}(w_1) = \{a_1, \ldots, a_{d_1}\}$, $S_2 = \mathrm{Supp}(w_2) = \{b_1, \ldots, b_{d_2}\}$, where $a_i \in V_1$, $b_j \in V_2$. Then the word with weight $d_1 d_2$ and support*

$$S = \{< a_i, b_j > | \ i = 1, \ldots d_1, j = 1, \ldots d_2\},$$

is in $C_2(\Gamma)^\perp$.

Proof: Let $X =< x, y >\in V_1 \times V_2$. If r_X meets v^S in a point $< a_i, b_j >\in V_1 \times V_2$, then either $x = a_i$ and $y \sim b_j$ or $x \sim a_i$ and $y = b_j$. Suppose the former. Then since $w_2 \in C_2(\Gamma_2)^\perp$, it meets r_y (from the adjacency matrix for Γ_2) evenly and thus there are an even number of points $< a_i, b >$ in r_X for $b \in S_2$. The same hold in the other case, and thus $v^S \in C_2(\Gamma)^\perp$. ■

Note 5 This can be extended in the obvious way to words in $C_2(\Box_{i=1}^n \Gamma_i)^\perp$ for n graphs Γ_i, giving a word of weight $\prod_{i=1}^n d_i$ from words of weight d_i in Γ_i, and in particular, if $\Gamma = \Gamma_i$ for all i, a word of weight d^n in $C_2(\Gamma^{\Box, n})$ from words of weight d in $C_2(\Gamma)$. In this case just one word suffices. Thus the minimum weight of $C_2(\Gamma^{\Box, n})$ is $\leq d^n$.

Lemma 5 *Let $\Gamma = \Gamma_1 \Box \Gamma_2$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, 2$. Suppose both $C_2(\Gamma_i)$ for $i = 1, 2$ are RLCD. Let $w_1 \in C_2(\Gamma_1)$ and $w_2 \in C_2(\Gamma_2)^\perp$, with $S_1 = \mathrm{Supp}(w_1) = \{a_1, \ldots, a_{d_1}\}$, $S_2 = \mathrm{Supp}(w_2) = \{b_1, \ldots, b_{d_2}\}$, where $a_i \in V_1$, $b_j \in V_2$. Then the word with weight $d_1 d_2$ and support*

$$S = \{< a_i, b_j > | \ i = 1, \ldots d_1, j = 1, \ldots d_2\},$$

is in $C_2(\Gamma)$.

Proof: Since $C_2(\Gamma)$ is *RLCD* we need only show that the inner product $(v^S, s_X) = 0$ for all $X \in V_1 \times V_2$. If A is an adjacency matrix for Γ, the row s_X of the matrix $A + I$ has 1's at X and at the neighbours of X. By abuse of language we can write, as explained in Notation 1, for $X =< x, y >$,

$$s_X = \{< x, y >\} \cup \{< x, y_i >| \; y_i \sim y\} \cup \{< x_i, y >| \; x_i \sim x\}. \qquad (5)$$

If $X =< a_i, b_j >$ then since $w_1 \in C_2(\Gamma_1) = C_2(R\Gamma_1)^\perp$, w_1 meets the row s_{a_i} of $A_1 + I$, where A_1 is an adjacency matrix for Γ_1, evenly, so s_X contains an even number of points of the the form $< a_k, b_j >$, including $k = i$. Since $w_2 \in C_2(R\Gamma_2)$, the row r_{b_j} of an adjacency matrix A_2 for Γ_2 meets w_2 evenly so v^S meets s_X in an even number of points of the form $< a_i, b_k >$ (where $k \neq j$). Thus $(v^S, s_X) = 0$.

If $X =< a_i, b >$ where $b \notin S_2$, then if $b \not\sim b_j$ for any $b_j \in S_2$, then s_X does not meet v^S at all. If $b \sim b_j$ for some j, then r_b meets w_2 evenly so there are an even number of points of the form $< a_i, b_j >$ in s_X.

If $X =< a, b_j >$ where $a \notin S_1$, then if $a \not\sim a_i$ for any $a_i \in S_1$, then s_X does not meet v^S at all. If $a \sim a_i$ for some i, then s_a meets w_1 evenly so there are an even number of points of the form $< a_i, b_j >$ in s_X.

If $X =< a, b >$ where $a \notin S_1$ and $b \notin S_2$, then s_X does not meet v^S at all, and the proof is complete. ■

The rank of the adjacency matrix from the Cartesian product is not given directly from the construction. However, we can use the eigenvalues of the graphs to get information regarding the possible dimension of the codes of the product graph. Since if λ is an eigenvalue for a matrix M then $\lambda + 1$ is an eigenvalue for $M + I$. If M is a $v \times v$ integral matrix with integral eigenvalues, then modulo p these will still be eigenvalues, but not necessarily all distinct. If none or at most one reduce to 0 modulo p then the p-rank of M will be v or $v - m_j$, respectively, where m_j is the multiplicity of the eigenvalue that is zero. In any case, the dimension of the zero eigenspace over $\mathbb{F}_p$ of the matrix A or $A + I$ is at most the sum m of the multiplicities of the eigenvalues that reduce to 0 modulo p, and thus the p-rank of A or $A + I$ is at least $\binom{n}{k} - m$.

This, together with the following result quoted in [2, Theorem 3], but due to [5], and also quoted in [25] allows one to get the 2-rank of the Cartesian product if the eigenvalues and multiplicities of all the constituents are known and integral, and if the constituents are all RLCD, using Result 4:

Result 8 *If $\Gamma = \Gamma_1 \Box \Gamma_2$, where $\Gamma_i = (V_i, E_i)$ with $|V_i| = N_i$ then if $\{\lambda_i \mid 1 \leq i \leq r\}$ are the eigenvalues of Γ_1, with multiplicities n_i, $1 \leq i \leq r$ and $\{\mu_i \mid 1 \leq i \leq s\}$ are the eigenvalues of Γ_2, with multiplicities m_i, $1 \leq i \leq s$ then the eigenvalues of Γ are $\{\lambda_i + \mu_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ with multiplicities $\{n_i m_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$.*

4 Direct (or Categorical) products of graphs $\Gamma_1 \times \Gamma_2$

If $\Gamma_i = (V_i, E_i)$, $i = 1, 2$, are graphs with $|V_i| = n_i$ and adjacency matrix A_i, then $\Gamma = \Gamma_1 \times \Gamma_2$ will denote the direct product of the graphs, with vertex set $V = V_1 \times V_2$. Here if $< x, y >, < u, v > \in V$ then

- adjacency is defined by $< x, y > \sim < u, v >$ in Γ if $x \sim u$ in Γ_1 and $y \sim v$ in Γ_2;
- if Γ_i are regular of valency ν_i, $i = 1, 2$, respectively, then $\Gamma_1 \times \Gamma_2$ is regular of valency $\nu_1\nu_2$;
- an adjacency matrix $A_{1\times 2}$ for $\Gamma_1 \times \Gamma_2$ is $A_{1\times 2} = A_1 \otimes A_2$;
- $\text{rank}(A_{1\times 2}) = \text{rank}(A_1)\text{rank}(A_2)$.

This last item follows from the properties of Kronecker products.

Proposition 2 *Let $\Gamma = \times_{i=1}^{n} \Gamma_i$. Then $C_2(\Gamma)$ is RLCD if all the $C_2(\Gamma_i)$ are RLCD for $i = 1, \ldots, n$. If p is an odd prime and the $C_p(\Gamma_i)$ are RLCD for $i = 1, \ldots, n$, then $C_p(\Gamma)$ is RLCD if n is odd.*

Proof: Consider $\Gamma_1 \times \Gamma_2$. If $A = A_{1\times 2} = A_1 \otimes A_2$, then $A^2 + A = A_1^2 \otimes A_2^2 + A_1 \otimes A_2$. Thus $A^2 + A = 0$ if $A_i^2 + A_i = 0$ for both i. This extends to a product of n graphs.

For $p > 2$, $(\otimes_{i=1}^{n} A_i)^2 = (\otimes_{i=1}^{n} A_i^2) = \otimes_{i=1}^{n}(-A_i) = -\otimes_{i=1}^{n} A_i$ if n is odd. ■

Corollary 2 *Let $\Gamma = \Gamma_1 \times \Gamma_2$. If $C_2(\Gamma)$ and $C_2(\Gamma_1)$ are RLCD, then $C_2(\Gamma_2)$ is RLCD.*

Proof: With notation as above, $0 = A^2 + A = A_1^2 \otimes A_2^2 + A_1 \otimes A_2 = A_1 \otimes A_2^2 + A_1 \otimes A_2 = A_1 \otimes (A_2^2 + A_2)$. Since this has rank 0, and A_1 does not have rank 0, we must have $A_2^2 + A_2 = 0$, so $C_2(\Gamma_2)$ is *RLCD*. ■

Note 6 If all the Γ_i are equal to a graph Γ then $\times_{i=1}^{n} \Gamma$ is written $\Gamma^{\times,n}$.

Lemma 6 *Let $\Gamma = \times_{i=1}^{n} \Gamma_i$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, \ldots, n$. Let $w_i \in C_2(\Gamma_i)^{\perp}$ be of weight d_i, with $S_i = \text{Supp}(w_i) = \{a_{i,1}, \ldots, a_{i,d_i}\}$, for $i = 1, \ldots, n$, where $a_{i,j} \in V_i$ for $j = 1, \ldots, d_i$. Then for any i and any choice of $b_j \in V_j$, $j = 1, \ldots, n$, $j \neq i$, the word with support*

$$S = \{< b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_n > \mid a \in S_i\}$$

is in $C_2(\Gamma)^{\perp}$ and has weight d_i. If there are m_i words of weight d_i in $C_2(\Gamma_i)^{\perp}$ then there are $m_i \prod_{j\neq i} |V_j|$ words of weight d_i of this form in $C_2(\Gamma)^{\perp}$, for each $i = 1, \ldots, n$.

If the Γ_i are all equal to a graph $\Lambda = (V, E)$, $d_i = d$, $m_i = m$, then the number of words of this form of weight d in $C_2(\Lambda^{\times,n})^{\perp}$ is $mn|V|^{n-1}$.

Proof: If $X =< x_1, \ldots, x_n > \in V_1 \times \ldots \times V_n$, then if v^S meets r_X, we must have $b_j \sim x_j$ for $j \neq i$, and $a \sim x_i$ for some $a \in S_i$. But since $w_i = v^{S_i} \in C_2(\Gamma_i)^{\perp}$ there must be an even number of such a, and thus v^S meets r_X evenly, showing that $v^S \in C_2(\Gamma)^{\perp}$. Counting the number of words is direct. ■

Corollary 3 *Let $\Gamma = \times_{i=1}^{n} \Gamma_i$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, \ldots, n$. Suppose $C_2(\Gamma_i)^\perp$ has minimum weight d_i for $i = 1, \ldots, n$, and let $d = \min\{d_i \mid i = 1, \ldots, n\}$. Then the minimum weight of $C_2(\Gamma)^\perp$ is at most d.*

Lemma 7 *Let $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, 2$. Suppose both $C_2(\Gamma_i)$ for $i = 1, 2$ are $RLCD$. Let $w_1 \in C_2(\Gamma_1)$ and $w_2 \in C_2(\Gamma_2)$, with $S_1 = \mathrm{Supp}(w_1) = \{a_1, \ldots, a_{d_1}\}$, $S_2 = \mathrm{Supp}(w_2) = \{b_1, \ldots, b_{d_2}\}$, where $a_i \in V_1$, $b_j \in V_2$. Then the word with weight $d_1 d_2$ and support*

$$S = \{< a_i, b_j >\mid i = 1, \ldots d_1, j = 1, \ldots d_2\},$$

is in $C_2(\Gamma)$.

Proof: Since $C_2(\Gamma)$ is $RLCD$ we need only show that $(v^S, s_X) = 0$ for all $X \in V_1 \times V_2$, where, for $X =< x, y >$,

$$s_X = \{< x, y >\} \cup \{< x_i, y_i >\mid x_i \sim x, y_i \sim y\}.$$

If $X =< a_i, b_j >$ then s_{a_i} meets w_1 evenly, so in a_i and an odd number of a_k, and likewise s_{b_j} meets w_2 evenly, so in b_j and an odd number of b_l. Thus s_X meets v^S in $< a_i, b_j >$ and an odd number of points $< a_k, b_l >$ where $a_k \sim a_i$ and $b_l \sim b_j$. Thus s_X meets v^S evenly.

If $X =< a_i, b >$, where $b \notin S_2$, then if $b \not\sim b_j$ for any b_j, s_X does not meet v^S at all. If $b \sim b_j$ for some j then since s_b meets w_2 evenly, i.e. it meets S_2 evenly. Since w_1 meets s_{a_i} evenly, there are an odd number of points in S_2 adjacent to a_i (excluding a_i). Thus counting the number of points of the form $< a_j, b_k >$ in s_X we get it to be even. The same argument works for a point X of the form $X =< a, b_j >$ where $a \notin S_1$.

If $X =< a, b >$ where $a \notin S_1, b \notin S_2$ then if either $a \not\sim a_i$ for any i or $b \not\sim b_j$ for any j, then s_X does not meet v^S at all. Thus suppose $a \sim a_i$ and $b \sim b_j$. Since s_a meets S_1 evenly and s_b meets S_2 evenly we certainly have the number of $< a_i, b_j >$ in s_x even.

This completes the proof. ■

Note 7 Lemma 7 extends to any number of graphs Γ_i for which $C_2(\Gamma_i)$ are all $RLCD$.

We can summarise these results concerning the parameters of the binary code of the direct product of graphs whose binary codes are $RLCD$:

Theorem 2 *Let $\Gamma = \times_{i=1}^{n} \Gamma_i$, where $\Gamma_i = (V_i, E_i)$ for $i = 1, \ldots, n$ and $C_2(\Gamma_i)$ is $RLCD$ for $i = 1, \ldots, n$. Suppose that $C_2(\Gamma_i)$ is a $[v_i, k_i, d_i]_2$ code and $C_2(\Gamma_i)^\perp$ a $[v_i, v_i - k_i, \delta_i]_2$ code, for $i = 1, \ldots, n$. Then*

1. *$C_2(\Gamma)$ is $RLCD$;*
2. *$C_2(\Gamma)$ is a $[\prod_{i=1}^{n} v_i, \prod_{i=1}^{n} k_i, d]_2$ code, where $d \leq \prod_{i=1}^{n} d_i$;*
3. *$C_2(\Gamma)^\perp$ is a $[\prod_{i=1}^{n} v_i, \prod_{i=1}^{n} v_i - \prod_{i=1}^{n} k_i, \delta]_2$ code, where $\delta = \min\{\delta_i \mid i = 1, \ldots, n\}$.*

Proof: Follows from Proposition 2, Lemma 6, Corollary 3, Lemma 7.■

For graphs Γ_i for $i = 1, \ldots, n$, and $\Gamma = \times_{i=1}^n \Gamma_i$, it is clear that if $\sigma_i \in \mathrm{Aut}(\Gamma_i)$ for $i = 1, \ldots, n$, then $\sigma = (\sigma_1, \ldots, \sigma_n)$ is in $\mathrm{Aut}(\Gamma)$, the action being defined in the obvious way.

Lemma 8 *If $C = C_2(\Gamma \times \Gamma)$ where $\Gamma = (V, E)$ with $|V| = n$, and I is an information set for $C_2(\Gamma)$, then an information set for C is $\mathcal{I} = I \times I = \{< i, j >| \, i, j \in I\}$.*

Furthermore, if S is an s-PD-set for $C_2(\Gamma)$ with information set I, then $S \times \{id_V\}$, or $\{id_V\} \times S$ is an s-PD-set for $\Gamma \times \Gamma$ with information set $\mathcal{I}$.

Proof: The proof is clear. ■

Note 8 Such information sets and s-PD-sets extend to a direct product of any number of graphs $\Gamma_i = (V_i, E_i)$, i.e. if $\Gamma = \times_{i=1}^n \Gamma_i$ and I_i is an information set for $C_2(\Gamma_i)$ then $I_1 \times \ldots \times I_n$ is an information set for $C_2(\Gamma)$. If S_i is an s_i-PD-set for $C_2(\Gamma_i)$, then for each $i = 1, \ldots, n$, the set

$$\{id_{V_1}\} \times \{id_{V_2}\} \times \ldots \times S_i \times \{id_{V_{i+1}}\} \times \ldots \times \{id_{V_n}\}$$

is an s_i-PD-set for $C_2(\Gamma)$.

5 Strong products of graphs $\Gamma_1 \boxtimes \Gamma_2$

If $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the strong product of the two graphs is the graph $\Gamma = \Gamma_1 \boxtimes \Gamma_2$ where $\Gamma = (V, E)$, $V = V_1 \times V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and for $< x, y >, < u, v > \in V$,

- adjacency is defined by $< x, y > \sim < u, v >$ in Γ if $x = u$ and $y \sim v$ in Γ_2, or $x \sim u$ in Γ_1 and $y = v$, or $x \sim u$ in Γ_1 and $y \sim v$ in Γ_2;
- if Γ_i is regular of valency ν_i then Γ is regular of valency $\nu_1 + \nu_2 + \nu_1\nu_2$;
- if A_1 is an adjacency matrix for Γ_1, and A_2 is an adjacency matrix for Γ_2, then an adjacency matrix for $\Gamma_1 \boxtimes \Gamma_2$ is

$$A = A_{1\boxtimes 2} = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 + A_1 \otimes A_2 = A_{1\square 2} + A_{1\times 2}.$$

Proposition 3 *Let $\Gamma = \boxtimes_{i=1}^n \Gamma_i$. Then $C_2(\Gamma)$ is RLCD if all the $C_2(\Gamma_i)$ are RLCD for $i = 1, \ldots, n$.*

Proof: Consider $\Gamma_1 \boxtimes \Gamma_2$. Then with notation as before

$$A = A_{1\boxtimes 2} = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 + A_1 \otimes A_2$$

So $A^2 = A_1^2 \otimes I_{n_2} + I_{n_1} \otimes A_2^2 + A_1^2 \otimes A_2^2 = A$, and thus $C_2(\Gamma)$ is also $RLCD$. Again this extends to the product of n graphs. ■

Corollary 4 *Let $\Gamma = \Gamma_1 \boxtimes \Gamma_2$. If $C_2(\Gamma)$ and $C_2(\Gamma_1)$ are $RLCD$, then $C_2(\Gamma_2)$ is $RLCD$.*

Proof: With notation as above, $A^2 = A_1^2 \otimes I_{n_2} + I_{n_1} \otimes A_2^2 + A_1^2 \otimes A_2^2 = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2^2 + A_1 \otimes A_2^2 = A = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 + A_1 \otimes A_2$, and so $(I_{n_1} + A_1) \otimes (A_2 + A_2^2) = 0$. Since $(I_{n_1} + A_1) \neq 0$, we must have $A_2 + A_2^2 = 0$ and thus $C_2(\Gamma_2)$ is $RLCD$. ■

Note 9 If all the Γ_i are equal to a graph Γ then $\boxtimes_{i=1}^{n} \Gamma$ is written $\Gamma^{\boxtimes,n}$.

6 Lexicographic products of graphs $\Gamma_1 \circ \Gamma_2$

If $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, $|V_1| = n_1$, $|V_2| = n_2$, and A_1, A_2 are adjacency matrices for Γ_1, Γ_2 respectively, the lexicographic product of the two graphs is the graph $\Gamma = \Gamma_1 \circ \Gamma_2$ where $\Gamma = (V, E)$, $V = V_1 \times V_2$, and, for $< x, y >, < u, v > \in V$

- adjacency is defined by $< x, y > \sim < u, v >$ in Γ if $x \sim u$ in Γ_1 or if $x = u$ and $y \sim v$ in Γ_2;
- if Γ_i is regular of valency ν_i then Γ is regular of valency $\nu_1 n_2 + \nu_2$;
- an adjacency matrix for Γ is

$$A = A_{1 \circ 2} = A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2.$$

It is easy to prove that the lexicographic product is associative.

Proposition 4 *Let $\Gamma = \Gamma_1 \circ \Gamma_2$. Then $C_2(\Gamma)$ is $RLCD$ if both $C_2(\Gamma_i)$ are $RLCD$ and n_2 is odd.*

Proof: For $A = A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2$, over $\mathbb{F}_2$,

$$A^2 = (A_1 \otimes J_{n_2})^2 + I_{n_1} \otimes A_2^2 = n_2 A_1^2 \otimes J_{n_2} + I_{n_1} \otimes A_2^2 = A$$

if n_2 is odd. ■

Corollary 5 *Let $\Gamma = \Gamma_1 \circ \Gamma_2$ and suppose $C_2(\Gamma)$ is $RLCD$. Then*

1. *if $C_2(\Gamma_1)$ is $RLCD$, then $C_2(\Gamma_2)$ is $RLCD$ if n_2 is odd;*
2. *if $C_2(\Gamma_2)$ is $RLCD$, then $C_2(\Gamma_1)$ is $RLCD$ if n_2 is odd.*

Proof: With notation as above, we assume

$$A^2 = n_2 A_1^2 \otimes J_{n_2} + I_{n_1} \otimes A_2^2 = A = A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2.$$

1. If $C_2(\Gamma_1)$ is $RLCD$ then $A_1^2 = A_1$, so we have $n_2 A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2^2 = A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2$, so $(n_2+1) A_1 \otimes J_{n_2} = I_{n_1} \otimes (A_2 + A_2^2)$. If n_2 is odd then we have $I_{n_1} \otimes (A_2 + A_2^2) = 0$, and hence $A_2 = A_2^2$ and $C_2(\Gamma_2)$ is $RLCD$.
2. If $C_2(\Gamma_2)$ is $RLCD$ then $A_2^2 = A_2$, so we have $n_2 A_1^2 \otimes J_{n_2} + I_{n_1} \otimes A_2 = A_1 \otimes J_{n_2} + I_{n_1} \otimes A_2$, so $(n_2 A_1^2 + A_1) \otimes J_{n_2} = 0$. If n_2 is even this would imply that $A_1 \otimes J_{n_2} = 0$, which is impossible, and so we must have n_2 odd, and $A_1 = A_1^2$, so $C_2(\Gamma_1)$ is $RLCD$. ■

Note 10 If all the Γ_i are equal to a graph Γ then $\circ_{i=1}^{n} \Gamma$ is written $\Gamma^{\circ,n}$.

7 Other products of graphs

7.1 Blackbox product of graphs $\Gamma_1 \blacksquare \Gamma_2$

If $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, $|V_1| = n_1$, $|V_2| = n_2$, and A_1, A_2 are adjacency matrices for Γ_1, Γ_2 respectively, the blackbox product of the two graphs is the graph $\Gamma = \Gamma_1 \blacksquare \Gamma_2$ where $\Gamma = (V, E)$, $V = V_1 \times V_2$, and for $< x, y >, < u, v > \in V$,

- adjacency is defined by $< x, y > \sim < u, v >$ in Γ if $x \sim u$ in Γ_1 or $y \sim v$ in Γ_2;
- if Γ_i is regular of valency ν_i then Γ is regular of valency $\nu_1 n_2 + \nu_2 n_1 - \nu_1 \nu_2$;
- an adjacency matrix for $\Gamma_1 \blacksquare \Gamma_2$ is

$$A = A_{1\blacksquare 2} = A_1 \otimes J_{n_2} + J_{n_1} \otimes A_2 - A_1 \otimes A_2.$$

Proposition 5 *Let $\Gamma = \blacksquare_{i=1}^{n} \Gamma_i$. Then $C_2(\Gamma)$ is RLCD if all the $C_2(\Gamma_i)$ are RLCD and all the n_i are odd, where $\Gamma_i = (V_i, E_i)$ and $|V_i| = n_i$.*

Proof: Let $\Gamma = \Gamma_1 \blacksquare \Gamma_2$. With notation as above, taking $n = 2$,

$$A^2 = n_2 A_1^2 \otimes J_{n_2} + n_1 J_{n_1} \otimes A_2^2 + A_1^2 \otimes A_2^2 = n_2 A_1 \otimes J_{n_2} + n_1 J_{n_1} \otimes A_2 + A_1 \otimes A_2 = A$$

since n_1, n_2 are odd. This extends to the blackbox product of n graphs. ■

Corollary 6 *Let $\Gamma = \Gamma_1 \blacksquare \Gamma_2$. If $C_2(\Gamma)$ and $C_2(\Gamma_1)$ are RLCD then if n_1 and n_2 are odd, $C_2(\Gamma_2)$ is RLCD.*

Proof: With notation as before,

$$A^2 = n_2 A_1^2 \otimes J_{n_2} + n_1 J_{n_1} \otimes A_2^2 + A_1^2 \otimes A_2^2 = A = A_1 \otimes J_{n_2} + J_{n_1} \otimes A_2 + A_1 \otimes A_2,$$

so $n_2 A_1 \otimes J_{n_2} + n_1 J_{n_1} \otimes A_2^2 + A_1 \otimes A_2^2 = A_1 \otimes J_{n_2} + J_{n_1} \otimes A_2 + A_1 \otimes A_2$, so since n_1, n_2 are odd, and thus $\equiv 1 \pmod 2$, we have $J_{n_1} \otimes A_2^2 + A_1 \otimes A_2^2 = J_{n_1} \otimes A_2 + A_1 \otimes A_2$, and hence $(J_{n_1} + A_1) \otimes (A_2^2 + A_2) = 0$. Since $A_1 \neq J_{n_1}$, we must have $A_2^2 + A_2 = 0$, so $C_2(\Gamma_2)$ is *RLCD*. ■

Note 11 If all the Γ_i are equal to a graph Γ then $\blacksquare_{i=1}^{n} \Gamma$ is written $\Gamma^{\blacksquare, n}$.

7.2 n-Multiples of a graph $n \otimes \Gamma$

If $\Gamma = (V, E)$, $\Omega = \{1, \ldots, n\}$, the n-multiple of Γ is the graph $n \otimes \Gamma = (V \times \Omega, E_n)$ with adjacency defined by $< x, i > \sim < y, j >$ if $x \sim y$ in Γ. If A is an adjacency matrix for Γ then an adjacency matrix for $n \otimes \Gamma$ is $A \otimes J_n$.

Proposition 6 *For $n \in Z$, if $C_p(\Gamma)$ is RLCD for some prime p, then $C_p(n \otimes \Gamma)$ is RLCD if $n \equiv -1 \pmod p$.*

Proof: Let A be an adjacency matrix for Γ. Then $(A \otimes J_n)^2 = A^2 \otimes J_n^2 = A^2 \otimes n J_n = (-A) \otimes n J_n = -A \otimes J_n$ if $n \equiv -1 \pmod p$. ■

7.3 n-Copies of a graph $n\Gamma$

If $\Gamma = (V,E)$, $\Omega = \{1,\dots,n\}$, then n-copies of Γ is the graph $n\Gamma = (V \times \Omega, E_n)$ with adjacency defined by $< x, i > \sim < y, j >$ if $x \sim y$ and $i = j$, or $x = y$ and $i \neq j$. If $|V| = v$, and Γ is regular of valency ν, then $n\Gamma$ is regular of valency $\nu + n - 1$. If A is an adjacency matrix for Γ then an adjacency matrix for $n\Gamma$ is $B = A \otimes I_n + I_v \otimes (J_n - I_n)$.

Proposition 7 *For $n \in Z$, if $C_2(\Gamma)$ is $RLCD$, then $C_2(n\Gamma)$ is $RLCD$ if n is odd.*

Proof: Let A be an adjacency matrix for Γ, B one for $n\Gamma$. Then over $\mathbb{F}_2$, $B^2 = (A \otimes I_n + I_v \otimes (J_n - I_n))^2 = A^2 \otimes I_n + I_v \otimes (J_n - I_n)^2 = A^2 \otimes I_n + I_v \otimes (nJ_n + I_n) = B$ if n is odd and $A^2 = A$, i.e. Γ is $RLCD$. ■

8 Examples

We will consider here the products of some graphs whose binary codes have been shown to be $RLCD$, in particular the triangular graphs $T(m)$, whose binary codes are shown in [16, Corollary 2] to be $RLCD$ for $m \geq 5$ odd, and the Paley graphs $P(q)$, for prime power $q \equiv 1 \pmod 8$, whose binary codes are also shown to be $RLCD$ in [16, Corollary 2].

First some background facts about these graphs and their binary codes.

1. Triangular graph $T(m) = \Gamma(m, 2, 1)$
   The triangular graph $T(m)$ has for vertices the set $V = \Omega^{\{2\}}$ where Ω is a set of size m, and $\{a,b\} \sim \{c,d\}$ if $|\{a,b\} \cap \{c,d\}| = 1$. It is a strongly regular graph with parameters $(\binom{m}{2}, 2(m-2), m-2, 4)$, and a uniform subset graph $\Gamma(m, 2, 1)$. The binary code from an adjacency matrix for $T(m)$ has been studied in various places (e.e. see [8,15]). If $m \geq 5$ is odd then $C = C_2(T(m))$ is a $[\binom{m}{2}, m-1, m-1]_2$ code and is $RLCD$, with $C^\perp = C_2(RT(m))$ a $[\binom{m}{2}, \binom{m-1}{2}, 3]_2$ code, with the words of weight 3 having the form $v^{\{a,b\}} + v^{\{a,c\}} + v^{\{b,c\}}$ for any set of three distinct elements of $\Omega = \{1,\dots,m\}$, $m \geq 5$ (see [15, Lemma 3.2]). For m even it is not $RLCD$.
   The symmetric group S_m is a subgroup of $\mathrm{Aut}(T(m))$.
   The eigenvalues of $\Gamma = T(m)$ are known since it is strongly regular. We will write λ_i^* for $\lambda_i + 1$, the eigenvalue of $R\Gamma$ (since the binary code is $RLCD$ for m odd) corresponding to the eigenvalue λ_i of Γ.
   - $\lambda_0 = 2m - 4$, $\lambda_0^* = 2m - 3$, $m_0 = 1$;
   - $\lambda_1 = m - 4$, $\lambda_1^* = m - 3$, $m_1 = m - 1$;
   - $\lambda_2 = -2$, $\lambda_2^* = -1$, $m_2 = \frac{1}{2}m(m-3)$;
2. Paley graph $P(q)$
   For q any prime power such that $q \equiv 1 \pmod 4$, $\Gamma = P(q)$ is the Paley graph on $\mathbb{F}_q$ with $x \sim y$ if and only if $x - y$ is a non-zero square. It is strongly regular with parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$. Its binary

code C is $RLCD$ if $q \equiv 1 \pmod 8$ and is a $[q, \frac{1}{2}(q-1), d]_2$ code, while RC is a $[q, \frac{1}{2}(q-1), d^\perp]_2$ code, where $d, d^\perp$ are not known in the general case. For any non-zero square $a \in F_q$ and any $b \in \mathbb{F}_q$, the map $\tau_{a,b} : x \mapsto ax + b$ is an automorphism of $P(q)$.
Again, the eigenvalues of $\Gamma = P(q)$ are known since it is strongly regular, and we write λ_i^* for $\lambda_i + 1$, the eigenvalue of $R\Gamma$ (since the binary code is $RLCD$ for $q \equiv 1 \pmod 8$), corresponding to the eigenvalue λ_i of Γ.

- $\lambda_0 = \frac{1}{2}(q-1)$, $\lambda_0^* = \frac{1}{2}(q+1)$, $m_0 = 1$;
- $\lambda_1 = \frac{1}{2}(-1+\sqrt{q})$, $\lambda_1^* = \frac{1}{2}(1+\sqrt{q})$, $m_1 = \frac{1}{2}(q-1)$;
- $\lambda_2 = \frac{1}{2}(-1-\sqrt{q})$, $\lambda_2^* = \frac{1}{2}(1-\sqrt{q})$, $m_2 = \frac{1}{2}(q-1)$.

8.1 Cartesian product

Recall from the definition at the beginning of Section 3 that for Γ_1 of valency ν_1 and Γ_2 of valency ν_2, the valency of $\Gamma_1 \Box \Gamma_2$ is $\nu_1 + \nu_2$. However we have no information in general for the rank of an adjacency matrix. If the eigenvalues of the graphs are known then the eigenvalues of $\Gamma_1 \Box \Gamma_2$ can be computed from Result 8.

(1) Cartesian product of n copies of $T(m)$

Recall that for $m \geq 5$ odd, $C_2(T(m))$ is a $[\binom{m}{2}, m-1, m-1]_2$ code and $C_2(T(m))^\perp$ is a $[\binom{m}{2}, \binom{m-1}{2}, 3]_2$ code. The valency of $T(m)$ is $2(m-2)$, so the valency of $\Box_{i=1}^n T(m) = T(m)^{\Box,n}$ is $2n(m-2)$. $C_2(T(m)^{\Box,n})^\perp$ has words of weight 3^n by Lemma 4, using words of weight 3 in $C_2(T(m))^\perp$.

We consider $\Gamma = \Box_{i=1}^n T(5) = T(5)^{\Box,n}$ with adjacency matrix A_n on 10^n vertices and valency $6n$. For A_1 and A_n we have, from Section 3,

$$A_1 = \begin{bmatrix} 0&1&1&1&1&1&1&0&0&0 \\ 1&0&1&1&1&0&0&1&1&0 \\ 1&1&0&1&0&1&0&1&0&1 \\ 1&1&1&0&0&0&1&0&1&1 \\ 1&1&0&0&0&1&1&1&1&0 \\ 1&0&1&0&1&0&1&1&0&1 \\ 1&0&0&1&1&1&0&0&1&1 \\ 0&1&1&0&1&1&0&0&1&1 \\ 0&1&0&1&1&0&1&1&0&1 \\ 0&0&1&1&0&1&1&1&1&0 \end{bmatrix}, \tag{6}$$

$$A_n = \begin{bmatrix} A_{n-1} & I & I & I & I & I & I & 0 & 0 & 0 \\ I & A_{n-1} & I & I & I & 0 & 0 & I & I & 0 \\ I & I & A_{n-1} & I & 0 & I & 0 & I & 0 & I \\ I & I & I & A_{n-1} & 0 & 0 & I & 0 & I & I \\ I & I & 0 & 0 & A_{n-1} & I & I & I & I & 0 \\ I & 0 & I & 0 & I & A_{n-1} & I & I & 0 & I \\ I & 0 & 0 & I & I & I & A_{n-1} & 0 & I & I \\ 0 & I & I & 0 & I & I & 0 & A_{n-1} & I & I \\ 0 & I & 0 & I & I & 0 & I & I & A_{n-1} & I \\ 0 & 0 & I & I & 0 & I & I & I & I & A_{n-1} \end{bmatrix},$$

where $I = I_{10^{n-1}}$, and A_n is $10^n \times 10^n$. By row reduction of A_n one can deduce that $\text{rank}_2(A_n) = 4 \times 10^{n-1} + 2 \times \text{rank}_2(A_{n-1})$; solving this recurrence and simplifying and using the fact that $\text{rank}_2(A_1) = 4$, gives

$$\text{rank}_2(A_n) = 2^{n-1}(5^n - 1).$$

The minimum weight of $C_2(T(5))$ is 4 and computation with Magma [4, 3] tells us that the minimum weight of $C = C_2(T(5)\Box T(5))$ is 12, and that of its dual is 9. Words of weight 12 in C can be constructed as is shown in Lemma 5 (and also the block $r_{<x,y>}$ has this weight), and of weight 9 in RC in Lemma 4, using words of weight 3 in $C_2(T(5))^{\perp}$. Thus $C = C_2(T(5)\Box T(5))$ is a $[100, 48, 12]_2$ code and $C^{\perp}$ is a $[100, 52, 9]_2$ code.

Using Results 4 and 8 and the known eigenvalues for $T(m)$ as quoted above one can deduce the following for $C = C_2(T(m)^{\Box,n})$:

- for $n = 1$, $\dim(C) = (m-1)$;
- for $n = 2$, $\dim(C) = (m-1)^2(m-2)$;
- for $n = 3$, $\dim(C) = \frac{1}{4}(m-1)^3(3m^2 - 12m + 16)$.

(2) Cartesian product of n copies of $P(q)$

The binary code of $P(q)$ is $RLCD$ if $q \equiv 1 \pmod 8$ and is a $[q, \frac{1}{2}(q-1), d]_2$ code, while RC is a $[q, \frac{1}{2}(q-1), d^{\perp}]_2$ code, where $d, d^{\perp}$ are not known in the general case. The valency is $\frac{q-1}{2}$. Thus the valency of $\Box_{i=1}^{n} P(q) = P(q)^{\Box,n}$ is $n\frac{q-1}{2}$.

For $q = 9$, $\Gamma = P(9)$ is strongly regular with parameters $(9, 4, 1, 2)$. Its binary code C is $RLCD$ and is a $[9, 4, 4]_2$ code, while RC is a $[9, 5, 3]_2$ code. If $\mathbb{F}_9$ has primitive element ω with minimal polynomial $X^2 + 2X + 2$, and the vertices are labelled by the sequence $[\omega^i \mid 0 \le i \le 8]$, let A_1 be an adjacency matrix for $P(9)$. Then $\Box_{i=1}^{n} P(9) = P(9)^{\Box,n}$ has adjacency matrix A_n on 9^n

vertices and valency $4n$, with A_1 and A_n given as follows, from Section 3:

$$A_1 = \begin{bmatrix} 0\,1\,1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,0\,1\,1\,1\,0\,0\,0 \\ 1\,0\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,0\,0\,0\,1 \\ 0\,1\,0\,1\,0\,0\,1\,1\,0 \\ 1\,1\,0\,0\,0\,0\,0\,1\,1 \\ 1\,0\,1\,0\,1\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1\,1\,1\,0\,1 \\ 0\,0\,1\,1\,0\,1\,0\,1\,0 \end{bmatrix},$$

$$A_n = \begin{bmatrix} A_{n-1} & I & I & 0 & 0 & I & I & 0 & 0 \\ I & A_{n-1} & 0 & I & I & I & 0 & 0 & 0 \\ I & 0 & A_{n-1} & I & 0 & 0 & I & 0 & I \\ 0 & I & I & A_{n-1} & I & 0 & 0 & 0 & I \\ 0 & I & 0 & I & A_{n-1} & 0 & I & I & 0 \\ I & I & 0 & 0 & 0 & A_{n-1} & 0 & I & I \\ I & 0 & I & 0 & I & 0 & A_{n-1} & I & 0 \\ 0 & 0 & 0 & 0 & I & I & I & A_{n-1} & I \\ 0 & 0 & I & I & 0 & I & 0 & I & A_{n-1} \end{bmatrix},$$

and where $I = I_{9^{n-1}}$, and A_{n-1} is $9^{n-1} \times 9^{n-1}$.

Computations with Magma show that $C_2(P(9)^{\square,2})$ has minimum weight 8 and its dual has minimum weight 9, the weight of rows in A_2 and $A_2 + I$ respectively.

By row reduction of A_n one can deduce that $\text{rank}_2(A_n) = 4 \times 9^{n-1} + \text{rank}_2(A_{n-1})$; solving this recurrence and simplifying and using the fact that $\text{rank}_2(A_1) = 4$, gives $\text{rank}_2(A_n) = \frac{1}{2}(9^n - 1)$, for $n \geq 1$.

Thus $C = C_2(P(9)^{\square,2})$ is a $[81, 40, 8]_2$ code and $C^\perp$ a $[81, 41, 9]_2$ code.

Using Results 4 and 8 and the known eigenvalues for $P(q^2)$ as quoted above, which are integral, one can deduce for $C = C_2(P(q^2)^{\square,m})$ for $1 \leq m \leq 3$, that $\dim(C) = \frac{1}{2}(q^{2m} - 1)$.

By computation, this formula also holds for $C = C_2(P(17)^{\square,m})$ for $1 \leq m \leq 3$, i.e. $\dim(C) = \frac{1}{2}(17^m - 1)$, where here the eigenvalues of $P(17)$ are not integers so the argument would not apparently apply.

(3) Cartesian product $T(m)\square P(q^2)$, $m \geq 5$ odd, $q^2 \equiv 1 \pmod 8$

Using Results 4 and 8 and the eigenvalues for $T(m)$ and $P(q^2)$, we have for $C = C_2(T(m)\square P(q^2))$, $\dim(C) = (m-1)(1+\frac{1}{4}(m(q^2-1)))$. Computationally with Magma we found that the minimum weight of C for $m = 5, q^2 = 9$ is 10, which is also the valency, and the minimum weight of $C^\perp$ is 9. Thus if $C = C_2(T(5)\square P(9^2))$, C is a $[810, 404, 10]_2$ code, and $C^\perp$ is a $[810, 396, 9]_2$ code.

Magma observations

- For $q \in \{9, 17, 25\}$ (i.e. $q \equiv 1 \pmod 8$), $\dim(C_2(P(q)\Box P(q))) = \frac{1}{2}(q^2 - 1)$, agrees with the formula proved for $q = 9$. Also $\dim(C_2(P(17)^{\Box,3})) = \frac{1}{2}(17^3 - 1)$.
- For $q = 5, 13$, $\dim(C_2(P(q)\Box P(q)) \neq \frac{1}{2}(q^2 - 1)$.
- $\dim(C_2(T(7)\Box T(7))) = 180$, while $\dim(C_2(T(7))) = 6$ does not fit the similar formula for $T(5)$.

8.2 Direct product

From the definition at the beginning of Section 4 we have that for Γ_1 of valency ν_1 and Γ_2 of valency ν_2, the valency of $\Gamma_1 \times \Gamma_2$ is $\nu_1\nu_2$. Furthermore, if A_1 is an adjacency matrix for Γ_1 and A_2 is an adjacency matrix for Γ_2, then the rank of an adjacency matrix for $\Gamma_1 \times \Gamma_2$ is $\text{rank}(A_1)\text{rank}(A_2)$.

(1) Direct product of n copies of $T(m)$, $m \geq 5$ odd

For $\Gamma = T(m)^{\times,n}$, Γ has valency $2^n(m-2)^n$ and an adjacency matrix has 2-rank $(m-1)^n$.

For example, for $m = 5$, $\times_{i=1}^n T(5) = T(5)^{\times,n}$ has adjacency matrix A_n, and valency 6^n, where for A_n we have, from Section 4, with A_1 is as in Equation (6):

$$A_n = \begin{bmatrix} 0 & A_{n-1} & A_{n-1} & A_{n-1} & A_{n-1} & A_{n-1} & A_{n-1} & 0 & 0 & 0 \\ A_{n-1} & 0 & A_{n-1} & A_{n-1} & A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} & 0 \\ A_{n-1} & A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} \\ A_{n-1} & A_{n-1} & A_{n-1} & 0 & 0 & 0 & A_{n-1} & 0 & A_{n-1} & A_{n-1} \\ A_{n-1} & A_{n-1} & 0 & 0 & 0 & A_{n-1} & A_{n-1} & A_{n-1} & A_{n-1} & 0 \\ A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} \\ A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} & A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} \\ 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} & A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} \\ 0 & A_{n-1} & 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} \\ 0 & 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} & A_{n-1} & A_{n-1} & A_{n-1} & 0 \end{bmatrix}. \quad (7)$$

Since $\text{rank}_2(A_n) = \text{rank}_2(A_1)^n = 4^n$, $C = C_2(T(5)^{\times,n})$ is a $[10^n, 4^n, d]_2$ code where $d \leq 6^n$. Further, $C^\perp$ is a $[10^n, 10^n - 4^n, 3]_2$ code, by Lemma 9 below.

The minimum weight of $C_2(T(5))$ is 4, and computation with Magma [4, 3] tells us that the minimum weight of $C = C_2(T(5) \times T(5))$ is 16, and that of its dual is 3.

Note 12 Words of weight 16 in C can be constructed as is shown in Lemma 7.

Lemma 9 *If $\Gamma = \times_{i=1}^n T(m) = T(m)^{\times,n}$, $m \geq 5$ odd, then the minimum weight of $C_2(\Gamma)^\perp = C_2(R\Gamma)$ is 3 and $n\binom{m}{2}^n \frac{m-2}{3}$ words of weight 3 have support of the form*

$$\{< x_1, \ldots, x_{n-1}, \{a, b\} >, < x_1, \ldots, x_{n-1}, \{a, c\} >, < x_1, \ldots, x_{n-1}, \{b, c\} >\},$$

where $a, b, c \in \{1, \ldots, m\}$ are distinct, the x_i are arbitrary vertices in $T(m)$, and the triple of vertices $\{a, b\}, \{a, c\}, \{b, c\}$ can be placed in any one of the n positions.

Proof: The minimum weight of $C_2(T(m))^\perp$ is 3 for all $m \geq 5$ and the words of weight 3 have support $\{\{a,b\},\{a,c\},\{b,c\}\}$ for the 3-sets in $\Omega = \{1,\ldots,m\}$, except for $n = 6$ when there are more: see [15]. A 3-set of the form shown is a special case of the word described in Lemma 6, and clearly the minimum weight cannot be less than 3. There are $\binom{m}{2}$ choices for each of the x_i, $\binom{m}{3}$ choices of the triples $\{a,b,c\}$, and n choices for the position of pairs from the triples, giving $n\binom{m}{2}^{n-1}\binom{m}{3} = n\binom{m}{2}^n \frac{m-2}{3}$ words of this form. ■

If an information set is known for $C_2(\Gamma)$ then we have an information set of $C_2(\Gamma \times \Gamma)$ from Lemma 8.

From [15, Theorem 1], taking only the case where $m \geq 5$ is odd, we have

Result 9 *Let $m \geq 5$ be odd, and $\mathcal{I} = \{\{1,m\},\{2,m\},\ldots,\{m-1,m\}\}$ and C denote a binary code of $T(m)$ with $\mathcal{I}$ in the first $m-1$ positions. Then C is a $[\binom{m}{2}, m-1, m-1]_2$ code with $\mathcal{I}$ as the information positions, and*

$$\mathcal{T} = \{id\} \cup \{(i,m) \mid 1 \leq i \leq m-1\}$$

is a PD-set for C of m elements in the symmetric group S_m acting on 2-sets, i.e. a $(\frac{m-3}{2})$-PD set for C of m elements.

Proposition 8 *For $m \geq 5$ odd, $C_2(T(m) \times T(m))$ has $(\frac{m-1}{2})$-PD-sets of size m, thus:*
for $I = \{\{1,i\} \mid 2 \leq i \leq m\}$, and information set $\mathcal{I} = \{< x,y >\mid x,y \in I\}$ for $C_2(T(m) \times T(m))$, the set $\mathcal{T} = \{((1,i),id) \mid 1 \leq i \leq m\}$ is a $(\frac{m-1}{2})$-PD-set for $C_2(T(m) \times T(m))$ of size m.

Proof: From [15] and Result 9, the set of vertices $I = \{\{1,i\} \mid 2 \leq i \leq m\}$ is an information set for $C_2(T(m))$ for m odd. Thus by Lemma 8 the set $\mathcal{I} = \{< x,y >\mid x,y \in I\}$ is an information set for $C_2(T(m) \times T(m))$. Let $\mathcal{C}$ denote the corresponding check symbols. Note that $\mathcal{I}$ consists of all the vertices in $T(m) \times T(m)$ that have a 1 in the 2-subset of $\Omega = \{1,\ldots,m\}$ in both positions.

The symmetric group S_m acts on $T(m)$ and thus $S_m \times S_m$ acts on $T(m) \times T(m)$. Let $\mathcal{T} = \{((1,i),id) \mid 1 \leq i \leq m\}$ acting on $T(m) \times T(m)$ where $(1,1)$ denotes the identity map on $T(m)$. Suppose a message arrives and $\frac{m-1}{2}$ errors occur. If they are all in $\mathcal{C}$ then (id,id) can be used, Otherwise, if at least one is in $\mathcal{I}$ then, since in these $\frac{m-1}{2}$ errors at most $m-1$ of the elements of $\Omega = \{1,\ldots,m\}$ appear in the 2-sets in the first coordinate, and one of them is 1, if j is the element that does not occur, then $j \neq 1$ and $((1,j),id)$ will move all the vertices into check. ■

Note 13 In [20] *PD*-sets for $C_2(T(m))$ of minimal size $\frac{m-1}{2}$ are shown to exist for $m \geq 5$ odd, i.e. for full error correction, with $t = \frac{m-3}{2}$.

Proposition 9 *For $m \geq 5$ odd, the minimum weight of $C_2(T(m) \times T(m))$ is $(m-1)^2$.*

Proof: For $m \geq 5$ odd, $C_2(T(m))$ is a $[\binom{m}{2}, m-1, m-1]_2$ code. $C_2(T(m))^\perp$ has minimum weight 3 and words of this weight have support $\{\{a,b\},\{a,c\},\{b,c\}\}$ for any choice of three distinct elements a,b,c from $\Omega = \{1,\ldots,m\}$, and these are the only words of weight 3. The number of such words containing a particular point $\{a,b\}$ in V, where $T(m) = (V,E)$, is clearly $m-2$.

Now consider $C = C_2(T(m) \times T(m))$. From Lemma 6, the minimum weight of $C^\perp$ is 3. We can label the $N = \binom{m}{2}$ vertices in V by $\{a_1,\ldots,a_N\}$, each a_i representing a 2-set from Ω. Let A_1 be the adjacency matrix for $T(m)$ with this labelling. The corresponding adjacency matrix for $T(m) \times T(m)$, since it is the Kronecker product $A_1 \otimes A_1$, will have A_1 in the positions in A_1 that have a 1, and the zero matrix 0 where there is a zero: see Equation (7). The vertices in $V \times V$ are labelled $< a_1, a_j >$ for $j = 1, \ldots N$ for the first set of N columns, then $< a_2, a_j >$ for $j = 1, \ldots N$ for the next, and so on. Writing $A_2 = [B_{i,j}]$ where the N^2 $N \times N$ matrices $B_{i,j}$ are either A_1 or 0, and $B_{i,j}$ has columns labelled by $\{< a_j, a_k >|\ k = 1,\ldots,N\}$ and rows by $\{< a_i, a_k >|\ k = 1,\ldots,N\}$. Clearly $B_{i,i} = 0$ for $1 \leq i \leq N$. Regarding A_2 as a block matrix, we can label the N rows by $\mathcal{R}_i$ for $i \in \{1,\ldots N\}$, and the N columns by $\mathcal{C}_j$ for $j \in \{1,\ldots N\}$. Thus $\mathcal{R}_i$ denotes the row of blocks $[B_{i,j} \mid 1 \leq j \leq N]$, and likewise $\mathcal{C}_j$ the column of blocks $[B_{i,j} \mid 1 \leq i \leq N]$.

From Lemma 7, C has words of weight $(m-1)^2$, so the minimum weight is at most $(m-1)^2$.

Let $w \in C$ and write $w = [w_1, w_2, \ldots, w_N]$ where w_j is the component from the blocks $B_{i,j}$ from $\mathcal{C}_j$, with support from the set of points $\{< a_j, a_1 >, \ldots, < a_j, a_N >\}$. We can suppose $w_1 \neq 0$. We know that $\mathrm{wt}(w_1) \geq m-1$. Suppose $\mathrm{Supp}(w_1) = \{< a_1, a_{j_1} >, \ldots, < a_1, a_{j_s} >\}$ where $s \geq m-1$. Every word of $C^\perp$ meets w evenly, and this is true for the weight-3 vectors with support $\{< a_1, a_{j_1} >, < b, a_{j_1} >, < c, a_{j_1} >\}$ where, if $a_1 = \{x,y\}$ then $b = \{x,z\}$, $c = \{y,z\}$, and x,y,z are distinct elements of Ω. Since there are $m-2$ distinct choices of z like this, there are at least another $m-2$ distinct points $< b, a_{j_1} >$ in $\mathrm{Supp}(w)$, so that the corresponding $m-2$ components w_i cannot be zero, and each of these w_i will have weight at least $m-1$. Thus $\mathrm{wt}(w) \geq (m-1)s \geq (m-1)^2$. ■

Note 14 A similar result appears to hold for $T(m) \times T(m)$ for m even but the same argument does not hold since it would only show the weight is at least $2(m-2)(m-1)$ instead of $4(m-2)^2$. Here the minimum weight for $C_2(T(m))$ is $2(m-2)$ and this was incorrectly stated, due to a typographical error, in [15, Theorem 1.1], although correctly stated in that paper in Result 3.

Corollary 7 *For $m \geq 5$ odd, $n \geq 1$, $C_2(T(m)^{\times,n})$ is a $[\binom{m}{2}^n, (m-1)^n, (m-1)^n]_2$ code and $C_2(T(m)^{\times,n})^\perp$ is a $[\binom{m}{2}^n, \binom{m}{2}^n - (m-1)^n, 3]_2$ code.*

Proof: Follows in the same way as in the proposition, and by induction. ■

(2) Direct product of n copies of $P(q)$, $q \equiv 1 \pmod 8$

For $\Gamma = \times_{i=1}^{n} P(q) = P(q)^{\times,n}$, the valency of Γ is $(\frac{q-1}{2})^n$, and the 2-rank of an adjacency matrix is also $(\frac{q-1}{2})^n$.

For $q = 9$, $\Gamma = \times_{i=1}^{n} P(9) = P(9)^{\times,n}$ has adjacency matrix A_n of 2-rank 4^n and valency 4^n. For A_n we have, from Section 4,

$$A_n = \begin{bmatrix} 0 & A_{n-1} & A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} & 0 & 0 \\ A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} & A_{n-1} & 0 & 0 & 0 \\ A_{n-1} & 0 & 0 & A_{n-1} & 0 & 0 & A_{n-1} & 0 & A_{n-1} \\ 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} & 0 & 0 & 0 & A_{n-1} \\ 0 & A_{n-1} & 0 & A_{n-1} & 0 & 0 & A_{n-1} & A_{n-1} & 0 \\ A_{n-1} & A_{n-1} & 0 & 0 & 0 & 0 & 0 & A_{n-1} & A_{n-1} \\ A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & 0 & 0 & A_{n-1} & 0 \\ 0 & 0 & 0 & 0 & A_{n-1} & A_{n-1} & A_{n-1} & 0 & A_{n-1} \\ 0 & 0 & A_{n-1} & A_{n-1} & 0 & A_{n-1} & 0 & A_{n-1} & 0 \end{bmatrix},$$

and where $I = I_{9^{n-1}}$, and A_{n-1} is $9^{n-1} \times 9^{n-1}$.

Since $C_2(P(9))^\perp$ has minimum weight 3, Lemma 6 gives words of this weight in $C_2(P(9)^{\times,n})^\perp$, and the minimum weight cannot be smaller than this. $C_2(P(9))$ has minimum weight 4, and Lemma 7 shows how words of weight 4^n can be constructed in $C_2(P(9)^{\times,n})$. This is also the valency of the graph. Thus $C_2((P(9)^{\times,n})$ is a $[9^n, 4^n, d]_2$ code where $d \leq 4^n$, and $C_2((P(9)^{\times,n})^\perp$ is a is a $[9^n, 9^n - 4^n, 3]_2$ code. For $n = 2$, computation with Magma shows that $d = 16$ is the minimum weight of $C_2(P(9) \times P(9))$.

Proposition 10 *For* $q \equiv 1 \pmod 8$*, and any information set* I *for* $C_2(P(q)$*,* $C_2(P(q) \times P(q))$ *has a 2-PD-set of size* q *for the information set* $\mathcal{I} = I \times I$ *given by*

$$S = \{(\tau_{1,a}, \tau_{1,0}) \mid a \in \mathbb{F}_q\}.$$

Proof: We have $\mathcal{I} = \{< x, y >| \, x, y \in I\}$. Let $C = \mathbb{F}_q \setminus I$, and $\mathcal{C} = \mathbb{F}_q \times \mathbb{F}_q \setminus \mathcal{I}$. If no errors occur then $(\tau_{1,0}, \tau_{1,0}) = id$ will work; if one error occurs in $\mathcal{C}$ then id will work. If one error occurs at $< x, y > \in \mathcal{I}$ then since $\{x + a \mid a \in \mathbb{F}_q\} = \mathbb{F}_q = I \cup C$, there exists $a \in F_q$ such that $x + a \in C$ and thus $< x, y >^{(\tau_{1,a}, \tau_{1,0})} = < x + a, y > \in \mathcal{C}$.

Now suppose two errors occur. If they are both in $\mathcal{C}$ then id can be used. If they are both in $\mathcal{I}$, suppose they are $< x_1, y_1 >, < x_2, y_2 > \in \mathcal{I}$. We wish to find $a \in \mathbb{F}_q$ such that $x_1 + a, x_2 + a \in C$. Let $S_1 = \{x_1 + a \mid a \in \mathbb{F}_q\}$, $S_2 = \{x_2 + a \mid a \in \mathbb{F}_q\}$, so $S_1 = S_2 = \mathbb{F}_q$. Let $S_1 \cap C = \{x_1 + e \mid e \in E\}$, of size $\frac{q+1}{2}$. Then $\{x_2 + e \mid e \in E\}$ has size $\frac{q+1}{2}$ and thus cannot be totally inside I. Thus there is an $e \in E$ such that $x_1 + e, x_2 + e \in C$, so $(\tau_{1,e}, \tau_{1,0})$ will move both points into check.

If one of the errors is in $\mathcal{I}$ and the other in $\mathcal{C}$, then suppose $< x_1, y_1 > \in \mathcal{I}$ and $< x_2, y_2 > \in \mathcal{C}$. Then we can look at the sets $S_1 = \{x_1 + a \mid a \in \mathbb{F}_q\}$, $S_2 = \{x_2 + a \mid a \in \mathbb{F}_q\}$, as before and by the same argument see that there is an $e \in E$ such that $x_1 + e, x_2 + e \in C$, so $(\tau_{1,e}, \tau_{1,0})$ will move $< x_1, y_1 >$ into check and keep $< x_2, y_2 >$ in check. ■

Note 15 This argument works for $q \equiv 1 \pmod 4$ also, when the code is not *RLCD*.

In [11] when $q \equiv 1 \pmod 8$ is a prime, 2-PD-sets of size 6 for $C_2(P(q))$ were given for an explicit information set using the fact that the code is cyclic:

Result 10 *[11, Corollary 2] Let $P(n)$ be the Paley graph of prime order n, where $n \equiv 1 \pmod 8$, and $C = [n, \frac{n-1}{2}]_p$ its code over $\mathbb{F}_p$ where p is a prime dividing $\frac{n-1}{4}$. For the information set for C given in $I = \{0, 1, \ldots, k-1\}$, where $k = \frac{n-1}{2}$, C has a 2-PD-set of size 6 as given by*

$$\{\tau_{1,b} \mid b \in \{0, k\}\} \cup \{\tau_{k,b} \mid b \in \{k, 2k, \frac{3k}{2}, \frac{k}{2} - 1\}\}.$$

Note 16 The smallest size a 2-PD-set for $C_2(P(q))$ can be is 4.

Corollary 8 *For $q \equiv 1 \pmod 8$ prime, information set $I = \{0, 1, \ldots, \frac{q-3}{2}\}$ for $C_2(P(q))$, $C_2(P(q) \times P(q))$ has a 2-PD-set of size 6 for the information set $\mathcal{I} = I \times I$, given by*

$$S = \{\tau_{1,b} \mid b \in \{0, \frac{q-1}{2}\}\} \cup \{\tau_{\frac{q-1}{2},b} \mid b \in \{\frac{q-1}{2}, q-1, \frac{3(q-1)}{2}, \frac{q-3}{2}\}\}.$$

8.3 Strong product of n copies of $T(5)$

Here we consider $\Gamma_n = \boxtimes_{i=1}^{n} T(5) = T(5)^{\boxtimes,n}$ with adjacency matrix A_n and valency $(1+\nu)^n - 1 = 7^n - 1$. For A_n we have, from Section 5,

$$\begin{bmatrix}
A_{n-1} & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & 0 & 0 & 0 \\
A_{n-1}+I & A_{n-1} & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & 0 & 0 & A_{n-1}+I & A_{n-1}+I & 0 \\
A_{n-1}+I & A_{n-1}+I & A_{n-1} & A_{n-1}+I & 0 & A_{n-1}+I & 0 & A_{n-1}+I & 0 & A_{n-1}+I \\
A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1} & 0 & 0 & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1}+I \\
A_{n-1}+I & 0 & 0 & 0 & A_{n-1} & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & 0 \\
A_{n-1}+I & 0 & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1} & A_{n-1}+I & A_{n-1}+I & 0 & A_{n-1}+I \\
A_{n-1}+I & 0 & 0 & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1} & 0 & A_{n-1}+I & A_{n-1}+I \\
0 & A_{n-1}+I & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1}+I & 0 & A_{n-1} & A_{n-1}+I & A_{n-1}+I \\
0 & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1}+I & A_{n-1} & A_{n-1}+I \\
0 & 0 & A_{n-1}+I & A_{n-1}+I & 0 & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}+I & A_{n-1}
\end{bmatrix}.$$

The minimum weight of $C_2(T(5))$ is 4, and computation with Magma [4,3] tells us that the minimum weight of $C = C_2(T(5) \boxtimes T(5))$ is 4, the minimum weight of its dual is 9, and $\mathrm{rank}_2(A_2) = 64$. Thus C is a $[100, 64, 4]_2$ code, and $C^\perp$ is a $[100, 36, 9]_2$ code.

8.4 Lexicographic product of $T(5)$ and K_3, $T(5) \circ K_3$

From Section 6, for $\Gamma_1 \circ \Gamma_2$ to be *RLCD* we need $\Gamma_2 = (V_2, E_2)$ where $|V_2| = n_2$ is odd. Let A be an adjacency matrix for $\Gamma = T(5) \circ K_3$, on 30 vertices, and valency 20, where K_n is the complete graph on n vertices. With $K = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ an adjacency matrix for K_3, and $J = J_3$, we have

$$A = \begin{bmatrix} K & J & J & J & J & J & J & 0 & 0 & 0 \\ J & K & J & J & J & 0 & 0 & J & J & 0 \\ J & J & K & J & 0 & J & 0 & J & 0 & J \\ J & J & J & K & 0 & 0 & J & 0 & J & J \\ J & J & 0 & 0 & K & J & J & J & J & 0 \\ J & 0 & J & 0 & J & K & J & J & 0 & J \\ J & 0 & 0 & J & J & J & K & 0 & J & J \\ 0 & J & J & 0 & J & J & 0 & K & J & J \\ 0 & J & 0 & J & J & 0 & J & J & K & J \\ 0 & 0 & J & J & 0 & J & J & J & J & K \end{bmatrix}.$$

Computation with Magma [4,3] tells us that $C_2(T(5) \circ K_3)$ is a $[30, 24, 2]_2$ code, with dual a $[30, 6, 9]_2$ code.

9 Conclusion

The main aim of the considerations in this research was to establish which of the types of products of graphs that have binary codes that are *RLCD*, have binary codes that are also *RLCD*. For those that satisfy this, the decoding method described in Section 2.2 from that developed in [17, Lemmas 1,2], can be used. Most of the products we studied did have this property, including the cartesian and direct products.

However, it also transpired that some of the graphs that have binary codes that can be decoded using permutation decoding, also allow permutation decoding of the product, specifically cartesian and direct products. Thus, from Lemma 8 we see that information sets for $C_2(\Gamma)$ immediately give information sets for the direct product and, furthermore, s-PD-sets for $C_2(\Gamma)$ can be used to define s-PD-sets for the direct product. Some examples of this are in Section 8.2. Note that this applies for codes that are not *RLCD* as well.

For the cartesian product, in [18,19] binary codes from the cartesian product of graphs $\mathcal{Q}_n^m$ (the m-ary n-cube), which are *LCD* but not *RLCD*, were shown for $n = 2$ and $m \geq 4$ to have s-PD-sets of minimal size (see Result 6), and up to the full error-correcting capability of the code in the case $n = 2$ and $m \geq 4$ even.

There is much scope here for further study.

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