Section 13.7: Cylindrical and Spherical Coordinates

1 Objectives

1. Convert coordinates from Cartesian to cylindrical and back. (5,7,9,11)

2. Convert coordinates from Cartesian to spherical and back. (15,17,19,21)

3. Convert quadric surfaces in cylindrical or spherical coordinates to Cartesian and identify. (31,32,33,34,35,36,38,42)

4. Draw solids bounded by quadric surfaces using Maple. (62,63)

2 Assignments

1. Read Section 13.7

2. Problems: 3,5,9,13,15,19,21,23,35,36,42,43,55,60,62

3. Read Section 14.2

3 Lecture Notes

Occasionally it helps in our understanding of equations to change coordinate systems. In this section, we’ll see that changing from rectangular to cylindrical or spherical coordinates helps to describe some surfaces in a much easier manner.

3.1 Cylindrical coordinates

The cylindrical coordinates of a point \( P(x, y, z) \) are \( P(r, \theta, z) \), where \( r \) and \( \theta \) are the polar coordinates you are familiar with from earlier calculus classes. The new feature in this class is the addition of the third dimension \( z \), but this \( z \) is the distance from the \( xy \)-plane to the point \( P \) in either coordinate system. That is, if \( z = c \) in the rectangular coordinate system, then \( z = c \) in the polar coordinate system as well. The values of \( r \) and \( \theta \) are the polar coordinates of the projection of \( P \) onto the \( xy \)-plane, so that \( r \) and \( \theta \) are just the polar coordinates of \( x \) and \( y \).

To translate from cylindrical \( P(r, \theta, z) \) to rectangular coordinates \( P(x, y, z) \), use the relationships

\[
x = r \cos \theta \\
y = r \sin \theta \\
z = z.
\]
To convert from rectangular coordinates $P(x, y, z)$ to cylindrical coordinates $P(r, \theta, z)$, use the relationships

\[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \]
\[ z = \frac{z}{z}. \]

What’s the whole point of doing this anyway? Whenever a given equation describes a surface in three dimensions with an axis of symmetry that is parallel to the $z$–axis, the same equation given in cylindrical coordinates is much easier to understand. In particular, the equation for a cylinder $x^2 + y^2 = c^2$ becomes $r = c$ in cylindrical coordinates (and hence the coordinate name). Note that since $r$ is a directed distance, it is allowed to attain both positive and negative values.

### 3.1.1 Example 1

The equation $z = r$ in cylindrical coordinates describes a cone in three dimensions. Why? Given that $r = x^2 + y^2$, we see that $z = x^2 + y^2$. Thus, the $z$– coordinate is equal to the radius of the circle with center at the origin in the $xy$–plane. When $z = c$, the surface in the $xy$–plane is a circle with radius $\sqrt{c}$. This sweeps out a cone in three dimensions.

If we allow $r$ to be a directed distance, then $r$ can attain negative values, and this then gives the cone below the $xy$–plane.

### 3.1.2 Example 2: Problem 13.7.10

To change $P(3, 3, -2)$ to cylindrical coordinates, note that $\tan \theta = \frac{3}{3} = 1$. Thus, $\theta = \frac{\pi}{4}$. We also have that $r^2 = x^2 + y^2 = 18$, so that $r = 3\sqrt{2}$.

Class Question:

1. Is there another set of cylindrical coordinates for this point?
2. What changes if we use $P(-3, -3, -2)$?

### 3.1.3 Example 3: Problem 13.7.50

We can also use the change in coordinates to simplify equations. Given $x^2 + y^2 - z^2 = 16$, we can rewrite this as

\[ r^2 \cos \theta + r^2 \sin \theta - z^2 = 16 \]
\[ \Rightarrow r^2 - z^2 = 16. \]

We recognize this equation as a hyperboloid from our work in Section 13.6; the equation representation in the cylindrical coordinate system should help to clarify this. In particular, note that $\theta$ is allowed to take on any value between 0 and $2\pi$. Also note that the minimum value for $|r|$ is 4, since $r^2 = 16 + z^2$. Thus, in the $xy$–plane, we should get a circle with radius 4, and the radius of the circle in the planes parallel to the $xy$–plane should be greater than 4.
3.2 Spherical Coordinates

A point in the spherical coordinate system is $P(\rho, \theta, \phi)$, where $\rho$ is the distance from the origin to the point $P(x, y, z)$. That is $\rho = |\vec{OP}|$, and hence $\rho \geq 0$ always. $\theta$ is still the angle of the projection of the point in the $xy$–plane, and $\phi$ is the angle between the point and the positive $z$–axis. Thus, $0 \leq \phi \leq \pi$, and $\theta$ is allowed to assume all values between 0 and $2\pi$.

As with the cylindrical coordinate system, the spherical coordinate system also simplifies certain types of equations very nicely. Given the name, you should guess that equations for spheres using this coordinate system reduce to constant equations. The equation of the sphere centered at the origin with radius $c$ is simply $\rho = c$ in the spherical coordinate system.

To translate from spherical to rectangular coordinates, we use the following relationships:

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi.
\end{align*}
\]

The argument that outlines this translation is given nicely in your text and won’t be repeated here. However, note that the text uses information about equality of alternate interior angles between parallel lines and the angle-side-angle theorem from geometry that gives congruence of triangles.

To translate from rectangular to spherical coordinates, we note that if we have the point $P(x, y, z)$, then $|\vec{OP}| = x^2 + y^2 + z^2$. Also, since $z = \rho \cos \phi$, we know that $\cos \phi = \frac{z}{\rho}$ so that we can recover $\phi$. The value for $\theta$ can then be determined using either the $x = \rho \sin \phi \cos \theta$ or $y = \rho \sin \phi \sin \theta$ relationships. Note that when translating from rectangular to spherical coordinates, you must calculate $\rho$ first, and then you must find $\phi$.

3.2.1 Example 4: Problem 13.7.22

In this problem, we want to find the spherical coordinates for $(-\sqrt{3}, -3, -2)$. We notes that $\rho^2 = 3 + 9 + 4$ so that $\rho = 4$. Thus $\cos \phi = \frac{-2}{4} = -0.5$, so that $\phi = \frac{2\pi}{3}$. Then using the equation for $y$ we get that $-3 = 4 \sin(\frac{2\pi}{3}) \sin \theta$, hence $\theta = \frac{4\pi}{3}$. 

3
3.2.2 Example 5: Problem 13.7.50 (again)

If we translate our earlier equation \( x^2 + y^2 - z^2 = 16 \) using spherical coordinates, we get that

\[
\begin{align*}
   x^2 + y^2 - z^2 &= 16 \\
   x^2 + y^2 + z^2 - 2z^2 &= 16 \\
   \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi - 2\rho^2 \cos^2 \phi &= 16 \\
   \rho^2 \sin^2 \phi(\cos^2 \theta + \sin^2 \theta) + \rho^2 \cos^2 \phi - 2\rho^2 \cos^2 \phi &= 16 \\
   \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi - 2\rho^2 \cos^2 \phi &= 16 \\
   \rho^2(\sin^2 \phi + \cos^2 \phi) - 2\rho^2 \cos^2 \phi &= 16 \\
   \rho^2 - 2\rho^2 \cos^2 \phi &= 16 \\
   \rho^2(1 - 2 \cos^2 \phi) &= 16.
\end{align*}
\]