Section 14.1: Vector Functions and Space Curves

1 Objectives

1. State and apply the definition of limit and continuity for vector-valued functions and identify the domain of a vector-valued function. (1,3,5)

2. Plot vector-valued functions using Maple and indicate their orientation. (7–12,13,17,19,23,25,31)

2 Assignments

1. Read Section 14.1

2. Problems: 1,5,6–12,14,17,21,29,31

3. Challenge: 36

4. Read Section 14.2

3 Maple Commands

We'll use the Maple command `spacecurve` to plot the spacecurves in this section. This command was also used in Section 13-5 when we looked at equations of lines in three dimensions.

4 Lecture Notes

Throughout this chapter, we'll begin to expand our knowledge of functions to vector spaces using our knowledge from previous calculus classes and the information from Chapter 13.

Recall that a function is a rule that assigns a value from a domain space onto a value from a range space. Thus far, we have only considered scalar-valued functions, that is, the range space consists of only scalar values. We want to extend our notion of functions to include **vector-valued** functions, which now have a range space consisting of vectors. Thus, given $t \in D$, where D represents the domain space, we return a vector based on the functional definition

$$\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

The vector function $\mathbf{r}(t)$ is parameterized by t; the values of the function are determined solely by the argument t and the individual function definitions $f(t)$, $g(t)$, and $h(t)$.

If $\mathbf{r}(t) =< f(t), g(t), h(t) >$, then

$$\lim_{t\to a}\mathbf{r}(t) =< \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) >,$$

provided each of the limits above exists.

The vector function $\mathbf{r}(t)$ is continuous at the point $t = a$ if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a).$$

Thus, $\mathbf{r}(t)$ is continuous at $t = a$ if and only if each of the functions $f(t)$, $g(t)$, and $h(t)$ are continuous at $t = a$.

4.0.1 Example 1: Problem 14.1.4

Given

$$\mathbf{r}(t) =< \frac{e^t - 1}{t}, \frac{\sqrt{1+t} - 1}{t}, \frac{3}{1+t} >,$$

does $\lim_{t\to 0}\mathbf{r}(t)$ exist? For this example, you need to apply L'Hopitals rule for evaluating limits of functions. You should get that

$$\lim_{t\to 0}\mathbf{r}(t) =< 1, \frac{1}{2}, 3 > .$$

4.1 Spacecurves

If $f(t)$, $g(t)$, and $h(t)$ are continuous on an interval I, then the set C of all points (x, y, z) in space such that

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

where t ranges over I is called a spacecurve. In general, the spacecurve is given as the set

$$C = \{(x, y, z) : x = f(t), y = g(t), z = h(t), t \in I\}.$$

The equation

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

are the parametric equations of C, where t is a parameter.

Spacecurves have direction, and can be thought of as the trace of a particle whose position at time t is $\mathbf{r}(t)$. The vector function $\mathbf{r}(t) =< f(t), g(t), h(t) >$ is the position vector of $P(f(t), g(t), h(t))$ on C. The direction of the spacecurve is determined by the path of the spacecurve as the values for t increase.

4.1.1 Example 2

From our knowledge of equations of lines, we know that $\mathbf{r}(t) =< 1+t, 2+5t, -1+6t >$ describes a spacecurve that is actually a line in three dimensions (notice that each of the terms in the vector function are linear).

4.1.2 Example 3

The spacecurve described by $\mathbf{r}(t) =< \cos t, \sin t, t >$ is a helix. The parametric equations are $x = \cos t$, $y = \sin t$, and $z = t$. The best way to visualize this spacecurve is to ignore the parametric equation for z at first. Thus we have $x = \cos t$, $y = \sin t$ which gives $x^2 + y^2 = 1$, the equation for a circle with radius 1. The only new adjustment is to allow z to vary, and the result of this is to pull the circle up in the $z-$axis, giving a helix structure to the spacecurve.
Class Question:

1. What is the spacecurve for $\mathbf{r}(t) =< \sin t, t, \cos t >$?

4.1.3 Example 4

Consider the equations $x^2+y^2 = 1$ and $y+z = 2$. What is the spacecurve determined by the intersection of these surfaces? Clearly, given the first equation, we note that we can parameterize x and y as $x = \cos t$ and $y = \sin t$. The circle is stretched at an angle by its intersection with the plane $y + z = 2$, or $z = 2 - \sin t$. Plot this spacecurve using Maple as

```
> spacecurve([cos(t),sin(t),2-sin(t)],t=0..2*Pi,axes=NORMAL);
```

4.1.4 Example 5: Problem 14.1.30

The vector function that represents the curve of intersection of these two surfaces can be found without too much trouble. Given $x^2 + y^2 = 4$, we know that $x = 2\cos t$ and $y = 2\sin t$ are two parameterizations of x and y. (Could we switch these?) Also, since $z = xy$, we have $z = 4\cos t \sin t$, or $z = 2\sin 2t$, using a trig identity. Thus, the vector function $\mathbf{r}(t)$ is

$$\mathbf{r}(t) =< 2\cos t, 2\sin t, 2\sin 2t > .$$