Section 15.7: Maximum and Minimum Values

1 Objectives

1. Find local extrema using the first and second derivative tests. (5,7,9,13,19,25)
2. Find absolute extrema for continuous functions defined on a closed and bounded set. (27,29,33,45)

2 Assignments

1. Read Section 15.7
2. Problems: 1,3,7,9,16,17,19,27,29,37,41,47
3. Challenge: 20,35,51
4. Read Section 15.8

3 Maple Commands

The most useful Maple command for this section will be the `solve` command. In order to find the critical points of the function $f(x, y) = y^2 - x^2$, you would first find the partial derivatives f_x and f_y using the Maple `diff` command. You then solve $f_x = 0$ and $f_y = 0$ using Maple. The following sequence will find critical points of $f(x, y)$ defined above.

```
> f:=y^2-x^2;
> fx:=diff(f,x);
> fy:=diff(f,y);
> solve({fx=0,fy=0},{x,y});
```

The last command solves the equations $f_x = 0$ and $f_y = 0$ and returns solutions for both x and y.

The following command list finds the 5 critical points of the function $f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$, labeled `cp1, cp2, cp3, cp4` and `cp5` .

```
> f := 10*x^2*y - 5*x^2 - 4*y^2 - x^4 - 2*y^4;
> fx := diff(f,x); fy := diff(f,y);
> cp1 := fsolve({fx=0,fy=0}, {x,y}, {x=-0.5..0.5,y=-0.5..0.5});
> cp2 := fsolve({fx=0,fy=0}, {x,y}, {x=2..3,y=1.5..2});
> cp3 := fsolve({fx=0,fy=0}, {x,y}, {x=-3..-2,y=1.5..2});
> cp4 := fsolve({fx=0,fy=0}, {x,y}, {x=0.5..1.5,y=0.5..1.5});
> cp5 := fsolve({fx=0,fy=0}, {x,y}, {x=-1.5..-0.5,y=0.5..1.5});
```

The second derivative test can be applied by differentiating f twice with respect to x and y, and finding f_{xy} as well. The following Maple commands find the discriminant function `d` and evaluate the discriminant function at each of the critical points. The second command `eval(fxx,cp1)` evaluates f_{xx} at the critical point so that you can determine if the critical point is a local minimum, a local maximum, or a saddle point.

```
> fxx := diff(f,x,x); d := diff(f,x,x)*diff(f,y,y) - diff(f,x,y)^2;
> eval(d, cp1); eval(fxx, cp1);
> eval(d, cp2); eval(fxx, cp2);
> eval(d, cp3); eval(fxx, cp3);
> eval(d, cp4); eval(fxx, cp4);
> eval(d, cp5); eval(fxx, cp5);
```

In this example, the first three points are points at which $f(x, y)$ obtains a local maximum value, and the last two points are saddle points for $f(x, y)$.

4 Lecture

In this section, we'll work on finding interior minima and maxima for continuous functions of two variables $f(x, y)$. A function of two variables $f(x, y)$ has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is "near" (a, b). $f(a, b)$ is called a local maximum. If $f(x, y) \geq f(a, b)$ when (x, y) is "near" (a, b), then $f(a, b)$ is a local minimum.

If $f(x, y) \leq f(a, b)$ ($f(x, y) \geq f(a, b)$) for all $(x, y) \in D$, where D is the domain of $f(x, y)$, then f has an absolute maximum (minimum) at (a, b).

If f has a local maximum or minimum at (a, b), and f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$. This means that $\nabla f(a, b) =< 0, 0 >$ and that the tangent plane is horizontal at the point (a, b). You can see that by remembering that the equation of the tangent plane to the surface $f(x, y)$ at the point (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z - z_0 = 0.$$

If $f_x(a, b) = f_y(a, b) = 0$, then the equation becomes $z = f(a, b)$.

4.1 Critical Points

The point (a, b) is a critical point of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if either f_x or f_y fails to exist at (a, b). If f has a local maximum or minimum at (a, b), then (a, b) is a critical point of the function. However, critical points are not necessarily local maxima or local minima.

4.1.1 Example 1: Problems 15.7.6 and 15.7.8

In 15.7.6, we have $f(x, y) = x^3y + 12x^2 - 8y$. Thus $f_x = 3x^2y + 24x$ and $f_y = x^3 - 8$. $f_y = 0$ when $x = 2$. Using this value, we solve for y in $f_x(2, y) = 3(4)y + 24(2) = 0$. This gives that $y = -4$, so the critical point is $(2, -4)$.

In 15.7.8, $f(x,y) = e^{4y-x^2-y^2}$. Thus $f_x = (-2x)e^{4y-x^2-y^2}$ and $f_y = (4-2y)e^{4y-x^2-y^2}$. Thus $f_x = 0$ when $x = 0$ (since the exponential function is never 0), and $f_y = 0$ when $y = 2$. This gives that $(0, 2)$ is a critical point of our function.

4.2 Classifying critical points

We can use the second derivative test to classify critical points.
Theorem: Suppose f_{xx}, f_{yy}, and f_{xy} ar continous on a disc centered at (a, b), and that (a, b) is a critical point of f. Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. Then:

1. if $D > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum;
2. if $D > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum;
3. if $D < 0$, $f(a, b)$ is a saddle point.

Note that if $D = 0$, our second derivative test tells us nothing and we must use other information about our function to classify the critical point.

4.2.1 Example 2: Revisit 15.7.6, 15.7.8

In 15.7.6, $f_{xx} = 6xy + 24$, $f_{yy} = 0$, and $f_{xy} = 3x^2$. Thus, $D(2, -4) = (-24)(0) - (12)^2 < 0$, so $(2, -4)$ is a saddle point.

In 15.7.8, $f_{xx} = (4x^2 - 2)e^{4y-x^2-y^2}$, $f_{yy} = (4y^2 - 16y + 14)e^{4y-x^2-y^2}$, and $f_{xy} = (-2x)(4 - 2y)e^{4y-x^2-y^2}$. Thus, $D(0, 2) = (-2e^4)(-2e^4) - 0 = 4e^4 > 0$, and since $f_{xx}(0, 2) < 0$, $(0, 2)$ is a local maximum.
Class Questions:

1. Since $(2, -4)$ in 15.7.6 is not a local maximum or a local minimum of f, where do the minima and maxima occur?

4.2.2 Example 3: Problem 15.7.10

Given $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$, we have four critical points: $(0, 0)$, $(-5/3, 0)$, $(-1, 2)$, and $(-1, -2)$. We also have $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, and $f_{xy} = 2y$. Thus, when we evaluate D at each point, we find that $D(0, 0) > 0$ and $f_{xx}(0, 0) > 0$ so $(0, 0)$ is a local minimum; $D(-5/3, 0) > 0$ and $f_{xx}(0, 0) < 0$ so $(-5/3, 0)$ is a local maximum; and that $D(-1, 2) < 0$ and $D(-1, -2) < 0$, so both of these latter points are saddle points.

4.2.3 Example 4: Problem 15.7.46

In this problem, we are asked to maximize the volume of a box subject to a constraint on the surface area. Thus, we want to maximize $V = lwh$ where $2lw+2wh+2lh = 64$. We can solve the last equation for h to get that $h = \dfrac{32 - lw}{w + l}$, and then find that

$V(l,w) = lw\dfrac{32 - lw}{w + l}$. I found that $l = \sqrt{32/3}$, $w = \sqrt{32/3}$, and $h = \sqrt{32/3}$. (Other critical points give that $l = 0$ or $w = 0$, which don't make sense.)

4.3 Absolute maximum and minimum

A set S in $I\!R^2$ is **closed** if it contains all of its boundary points. That is, the set $\{(x,y) : x^2 + y^2 < 4\}$ is open, but $\{(x,y) : x^2 + y^2 \leq 4\}$ is closed.

A set is **bounded** if I can enclose all members of the set in a circle of radius $\delta > 0$. Thus, a set S is bounded if $S \subset \{(x,y) : x^2 + y^2 < \delta\}$ for some $\delta > 0$.

This gives rise to the extreme value theorem:

Theorem: If f is continuous on a closed, bounded set $D \subset I\!R^2$, then f attains a minimum value $f(x_1, y_1)$ and a maximum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

This means that we can find an absolute minimum and an absolute maximum for our function $f(x,y)$ as long as our domain set D is closed and bounded. In order to find the absolute minimum and maximum, do the following:

1. Find and classify critical points of f in D by solving $f_x = 0$ and $f_y = 0$ and evaluating the discriminant function.

2. Find the extreme values of f on the boundary of D. You must evaluate your function along the boundary regions, making sure to evaluate the function at endpoints of your domain when necessary. If the boundary of your domain is a circle, for instance, you have no endpoints to be concerned with.

3. The largest value from the steps above is the absolute maximum; the smallest value is the absolute minimum.

4.3.1 Example 5: Problem 15.7.28

To find the absolute maximum and minimum of $f(x,y) = x^2 + 2xy + 3y^2$, we need to find the critical points and then evaluate along the boundary of the triangular domain. This means that we need to evaluate the function along the lines $y = 1$, $x = -1$, and $y = x - 1$. (Draw a picture of the domain to see this.) We'll also need to evaluate the function at the corner points of the domain. The critical point of the function is $(0,0)$, and $f(0,0) = 0$. Along the line $y = 1$, $f(x,1) = x^2 + 2x + 3$, and $-1 \leq x \leq 2$. There is a maximum at $x = 2$ and a minimum at $x = -1$. $f(2,1) = 11$, and $f(-1,1) = 2$. Similarly, $f(-1,y) = 1 - 2y + 3y^2$, and here $-2 \leq y \leq 1$. This has a maximum at $y = -2$ and a minimum at $y = 1/3$, where $f(-1,-2) = 17$ and $f(-1, 1/3) = 2/3$. Finally, along the line $y = x - 1$, we have that $f(x, x-1) = 6x^2 - 8x + 3$, where $-1 \leq x \leq 2$. The maximum along this line occurs when $x = -1$ and the minimum occurs when $x = 2/3$, where $f(-1,-2) = 17$ and $f(2/3, -1/3) = 1/3$.

Thus, the absolute maximum of f on our domain D is $f(-1,-2) = 17$ and the absolute minimum is $f(0,0) = 0$.

Class Questions:

1. What changes when you extend the boundary lines?