Section 17.8: Stokes Theorem

1 Objectives

1. Use Stokes' Theorem to evaluate line and surface integrals.

2 Assignments

1. Read Section 17.8
2. Problems: 1,3,7
3. Challenge: 5,13

3 Maple Commands

4 Lecture

Stokes' Theorem is really an extension or variant of Green's Theorem. Stoke's Theorem relates a surface integral over S to a line integral around the boundary of S. The orientation of S needs to induce the positive orientation of C, or ∂S, the boundary of S. This means that if you walk in the positive direction around the boundary region, labeled as both C and ∂S in your text, then S is always on your left.

You want to use Stokes' Theorem for the same reasons that you might want to use Green's Theorem. Some integrals are easier to evaluate as line integrals instead of surface integrals, and vice-versa.

4.1 Stokes' Theorem

The theorem is the following:
Theorem: Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Let's work two examples using this theorem.

4.1.1 Example 1: Problem 17.8.4

We are given $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$, and we want to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. If we use Stokes' Theorem, we see that we can rewrite this as

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \end{aligned}$$

The surface S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward. The boundary curve C is the the circle $x^2 + y^2 = 4$ at $z = 5$, so the parametric representation is $< 2\cos(t), 2\sin(t), 5 >$. (We can't set $z = 0$ since this boundary lies above the $xy-$plane. But the $z-$ coordinate is constant at $z = 5$.)

Thus, the surface integral can be evaluated as a line integral as

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} < (2\sin(t))(5), (2\cos(t))(5), (2\cos(t))(2\sin(t)) > \cdot < -2\sin(t), 2\cos(t), 0 > \; dt \\ &= \int_0^{2\pi} -20\sin^2(t) + 20\cos^2(t)\, dt \\ &= 20 \int_0^{2\pi} \cos(2t)\, dt \\ &= 0. \end{aligned}$$

Class Questions:

1. Can you evaluate the surface integral to check this answer?
2. What changes if the surface is the part of the paraboloid lying above $z = 8$?

4.1.2 Example 2: Problem 17.8.8

In this problem, we are given $\mathbf{F} = e^{-x}\mathbf{i} + e^{x}\mathbf{j} + e^{z}\mathbf{k}$, and we want to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant.

The curve C is only piecewise smooth, and we would need to separate C into its smooth components before evaluating the line integral. Or, we could use Stokes' Theorem, since the surface in this case is easy to represent.

The *curl* of $\mathbf{F}$ is $e^x\mathbf{k}$. Thus, using Stokes' Theorem, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ &= \iint_S e^x \mathbf{k} \cdot d\mathbf{S}. \end{aligned}$$

The surface S is $z = 1 - 1/2y - x$, so the surface integral becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA,$$

where in this case $P = 0$, $Q = 0$, and $R = e^x$. Also, given that $2x + y + 2z = 2$, we know that $0 \leq x \leq 2$ and $0 \leq y \leq 2 - 2x$ in the $xy-$plane, giving the limits of integration on D.

Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_D ((0)(-1) - (0)(-1/2) + e^x)\ dA \\
&= \int_0^1 \int_0^{2-2x} e^x\, dy\, dx \\
&= \int_0^1 ye^x|_0^{2-2x}\ dx \\
&= \int_0^1 (2 - 2x)e^x\, dx \\
&= 2e - 4.
\end{aligned}$$

Class Problems:

1. Fill in the blanks between the last two lines above.
2. Try to draw C to determine how difficult it would be to represent the line integral.
3. Can you evaluate the line integral to check this answer?
4. What changes if the curve is the boundary of $2x+2y+z = 2$ in the first octant?

4.2 Extra notes

As long as two oriented surfaces have the same boundary C, then the surface integrals are the same.

Also, note that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds.$$

$\mathbf{v} \cdot \mathbf{T}$ is the component of $\mathbf{v}$ in the direction of $\mathbf{T}$, the unit tangent vector. The closer the direction of $\mathbf{v}$ is to $\mathbf{T}$, the larger is the value of $\mathbf{v} \cdot \mathbf{T}$. The line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$ measures the tendency of a fluid to move around C and is the circulation of $\mathbf{v}$ around C. Thus, $(\nabla \times \mathbf{v}) \cdot \mathbf{n}$ measures the rotating effect of a fluid about $\mathbf{n}$, since the *curl* of a vector tells us something about the axis of rotation for a fluid with velocity field $\mathbf{v}$.