Read. *Mathematical Concepts in Modern Biology*, Chapter 1: Mechanisms of gene regulation: Boolean network models of the lactose operon in *Escherichia coli*, by R. Robeva, B. Kirkwood, and R. Davis, pages 1–35.

Do. Familiarize yourself with either Macaulay2, Singular, or Sage.

Exercises.

1. Given a Boolean network $(f_1, f_2, f_3, f_4, f_5)$, suppose that $\{x_1+x_5,\ x_3+1,\ x_2+x_5+1,\ x_4\}$ is a Gröbner basis of the ideal $I = \langle f_i + x_i \mid i = 1, \dots, 5\rangle$. What can you deduce from this? Your answer should consist of several clear and complete sentences, and *you should be as specific as possible*.

2. Consider the following system of polynomial equations:

$$\begin{aligned} x^2 + y^2 + xyz &= 1 \\ x^2 + y + z^2 &= 0 \\ x - z &= 0 \end{aligned}$$

   (a) Use a software package to compute a Gröbner basis for this system over $\mathbb{Q}$. Include a print-out of the code that you used.

   (b) Use the Gröbner basis you just computed to write a simpler system of polynomial equations that has the same set of solutions. Solve that system *by hand* (it's not hard) to find all (complex-valued) solutions to the original system.

   (c) Repeat the previous two parts, but over the binary field, $\mathbb{F}_2 = \{0, 1\}$.

   (d) Now, do Parts (a) and (b) but over the ternary field, $\mathbb{F}_3 = \{0, 1, 2\}$.

3. Consider the following simple model of the *lac* operon:

$$\begin{aligned} f_M &= \overline{R} & \qquad f_R &= \overline{A} \\ f_P &= M & \qquad f_A &= L \wedge B \\ f_B &= M & \qquad f_L &= P \end{aligned}$$

   For this problem, make the convention that $(x_1, x_2, x_3, x_4, x_5, x_6) = (M, P, B, R, A, L)$.

   (a) Sketch the wiring diagram of this Boolean network.

   (b) Justify each function in a single sentence. What other assumptions are made in this model? (E.g., presence or absense of external lactose and glucose?)

   (c) Write each function as a polynomial over $\mathbb{F}_2 = \{0, 1\}$. Then, write out a system of equations whose solutions are the fixed points of the Boolean network.

   (d) Find the fixed points by computing a Gröbner bases with a software package.

   (e) Compute and print out the entire phase space of your model with the help of the Cyclone algorithm on AlgoRun, at `http://algorun.org/browse`.

4. Consider the following Boolean network:

$$(f_1, f_2, f_3) = \big(\overline{x_1} \wedge x_2 \wedge \overline{x_3},\ (x_1 \wedge \overline{x_3}) \vee (\overline{x_1} \wedge x_3),\ \overline{x_1} \wedge x_2 \wedge \overline{x_3}\big).$$

   (a) Sketch the wiring diagram of this Boolean network.

   (b) Write these functions as polynomials in $\mathbb{F}_2[x_1, x_2, x_3]/\langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3\rangle$. Feel free to use Macaulay2, Sage, or Singular.

   (c) Sketch the (synchronous) phase space of this Boolean network. Feel free to use Algorun.

   (d) Sketch the asynchronous phase space of this Boolean network, and find all strongly connected components.

   (e) Classify the nodes of the asynchronous phase space as transient points, cyclic attractors, or complex attractors.

5.